

GRADED QUOTIENTS OF RAMIFICATION GROUPS OF LOCAL FIELDS WITH IMPERFECT RESIDUE FIELDS

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ABSTRACT. We prove that the graded quotients of the filtration by ramification groups of any henselian discrete valuation field of residue characteristic $p > 0$ are \mathbf{F}_p -vector spaces. We define an injection of the character group of each graded quotient to a twisted cotangent space defined using a cotangent complex.

Let K be a henselian discrete valuation field of residue characteristic $p > 0$ and L be a finite Galois extension of K of Galois group G . Then, the decreasing filtration on G by upper numbering ramification groups $G^r \subset G$ is defined in [1], using rigid geometry. The definition is rephrased purely in the language of schemes in [18] and is briefly recalled in Sections 3.3 and 4.1. The ramification groups G^r are normal subgroups indexed by positive rational numbers $r > 0$.

In this article, we study the graded quotients $\mathrm{Gr}^r G = G^r/G^{r+}$ where $G^{r+} = \bigcup_{s>r} G^s \subset G^r$. The inertia subgroup $I \subset G$ and its p -Sylow subgroup $P \subset I$ equal $G^{1+} \subset G^1$ [1, Proposition 3.7 (1)] and we have $\#G^1 = \prod_{r \geq 1} \#\mathrm{Gr}^r G$ [1, Theorem 3.8]. We prove the following theorem.

Theorem 4.3.1. 1. *For $r > 1$, the graded quotients $\mathrm{Gr}^r G = G^r/G^{r+}$ are \mathbf{F}_p -vector spaces.*

2. *Let C be a field of characteristic different from p and let V be a representation of G on a C -vector space of finite dimension. Then, the total dimension $\dim_{\mathrm{tot}}(V)$ is an integer.*

In the classical case where the residue field is perfect, Theorem 4.3.1.1 is proved in [22, Corollaire 1 of Proposition 7, Section 2, Chapitre IV] by using the lower numbering filtration. In this case, the total dimension $\dim_{\mathrm{tot}}(V)$ equals the sum of $\dim V$ and the Swan conductor $\mathrm{Sw}(V)$ and Theorem 4.3.1.2 follows from the Hasse–Arf theorem [22, Chap. V §7 Théorème 1].

It is proved in [2] that the graded quotients are abelian under the assumption that the residue characteristic $p > 0$ is not a uniformizer or for a logarithmic variant of the filtration. For the logarithmic variant, Theorem 4.3.1.1 is proved and a counterpart of the injection (4.20) below is constructed in the equal characteristic case in [14] by a geometric method and in the mixed characteristic case in [15]. For the original non-logarithmic filtration, Theorem 4.3.1.1 is proved and the injection (4.20) below is constructed in the equal characteristic case in [17] by a geometric method similar to that in [14]. Thus Theorem 4.3.1.1 and the construction of the injection (4.20) remained open in the mixed characteristic case.

By a different method using p -adic differential equations, Xiao Liang proved both Theorem 4.3.1.1 and 2 in the equal characteristic case in [23] and in the mixed characteristic in [24] under certain mild assumptions.

The characteristic cycle of an étale sheaf on a smooth scheme over a perfect field of characteristic $p > 0$ is computed using the injection (4.20) on a neighborhood of the generic point of the ramification divisor in [16]. We expect that the injection (4.20) in the mixed characteristic case enables to compute the characteristic cycle of a constructible sheaf in mixed characteristic case in the framework of [20] as well.

We give two proofs of Theorem 4.3.1.1. The first proof is by the reduction to the classical case. A similar approach was proposed in [5]. Using the first proof, we give in [19] a characterization of the filtration by ramification groups by a certain functoriality and the classical case where the residue field is perfect. We also deduce Theorem 4.3.1.2 from the classical case.

The second proof is more geometric and by the reduction to the case where r is an integer. In the notation prepared below, the second proof gives rise to an injection

$$(4.20) \quad \text{ch}: \text{Hom}(\text{Gr}^r G, \mathbf{F}_p) \rightarrow \text{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+}, H_1(L_{\bar{F}/S}))$$

for every rational number $r > 1$, called the characteristic form. The morphism (4.20) is a generalization of a non-logarithmic variant of the refined Swan conductor defined by Kato [13, Corollary (5.2)] in the case where G is abelian.

The two proofs are based on the following idea. For a discrete valuation field K , suppose that we find an extension K' such that the induced morphisms on the graded quotients of ramification groups be isomorphisms. Then, the conclusion of Theorem 4.3.1 for K' would imply the same conclusion for K . In the first proof, we find K' with perfect residue field where the conclusion is a classical result. In the second proof, we find K' such that the index r' corresponding to a given rational number $r > 1$ is an integer. For integral indices, we prove that a geometric construction similar to the equal characteristic case in [17] yields Theorem 4.3.1 and an injection (4.20). Finding such K' is thus a crucial reduction step in the proof of Theorem 4.3.1.

To establish the reduction step, we need to go back to a geometric construction behind the definition of ramification groups recalled in Section 3.3. The construction provides a certain finite Galois covering of a vector space $\Theta^{(r)}$ over an algebraic closure of the residue field such that the Galois group is the graded quotient $\text{Gr}^r G$. Since $\text{Gr}^r G$ is a p -group, if the induced morphism on $\Theta^{(r)}$ induced by $K \rightarrow K'$ is dominant, the surjectivity of the morphism on the fundamental groups proved in Section 2.1 induces a surjection on the graded quotients as required. The necessary functorial properties of the vector space $\Theta^{(r)}$ are prepared in Section 1 by studying certain normal sheaves and modules of differentials.

The key point in the second proof in the case where $r > 1$ is an integer is the property that the Galois covering of $\Theta^{(r)}$ of the Galois group $\text{Gr}^r G$

carries a structure of smooth group scheme such that the covering is a morphism of group schemes. This is proved by applying a general criterion prepared in Section 2.2 to the geometric construction similar to that in [17]. The construction is a refinement of that in [15] and is given in Section 3.4. Further applying the description of extension groups of a vector space over a field of characteristic $p > 0$ by \mathbf{F}_p in Section 2.1, we define the injection (4.20).

We sketch the idea more precisely. Both the first proof of Theorem 4.3.1 and the construction of (4.20) are based on the construction of the tangent space of a local ring at a geometric closed point defined by an algebraic closure of the residue field. The construction is globalized in [21] using a certain universality. Let $S = \text{Spec } \mathcal{O}_K$ and \bar{F} be an algebraic closure of the residue field $F = \mathcal{O}_K/\mathfrak{m}_K$. We show in Proposition 1.1.3 that the cotangent complex $L_{\bar{F}/S}$ is acyclic except at degree -1 and the cohomology group $H_1(L_{\bar{F}/S})$ is an \bar{F} -vector space fitting in an exact sequence

$$(0.1) \quad 0 \rightarrow \mathfrak{m}_K/\mathfrak{m}_K^2 \otimes_{\mathbf{F}_p} \bar{F} \rightarrow H_1(L_{\bar{F}/S}) \rightarrow \Omega_{\bar{F}/\mathbf{F}_p}^1 \otimes_{\mathbf{F}_p} \bar{F} \rightarrow 0.$$

We define the tangent space of $S = \text{Spec } \mathcal{O}_K$ at the geometric closed point $\text{Spec } \bar{F}$ as the spectrum

$$(0.2) \quad \Theta_{K, \bar{F}} = \text{Spec } S(H_1(L_{\bar{F}/S}))$$

of the symmetric algebra over \bar{F} . If $\mathcal{O}_{K_0} \subset \mathcal{O}_K$ is a discrete valuation subring with perfect residue field k , we have a canonical surjection

$$(0.3) \quad H_1(L_{\bar{F}/S}) \rightarrow \Omega_{\mathcal{O}_K/\mathcal{O}_{K_0}}^1 \otimes_{\mathcal{O}_K} \bar{F}$$

and (0.3) is an isomorphism if the ramification index e_{K/K_0} is not 1, by Proposition 1.1.5. The exact sequence (0.1) is canonically identified with the scalar extension of that constructed in [8, Proposition 9.6.14] by a different method ([21, Corollary 2.7, Corollary 4.12]).

We say that an extension K' of K of discrete valuation fields is dominant on the tangent spaces, or tangentially dominant, if the induced morphism $S(H_1(L_{\bar{F}/S})) \rightarrow S(H_1(L_{\bar{F}'/S'}))$ of symmetric algebras is an injection. We derive Theorem 4.3.1 from the following fact by constructing a tangentially dominant extension K' of K with perfect residue field in Proposition 1.1.10.

Corollary 4.2.6 (1) *Let K be a henselian discrete valuation field and let L be a finite Galois extension of K of Galois group G . Let K' be a tangentially dominant extension of henselian discrete valuation field, let $L' = LK'$ be a composition field and let G' be the Galois group. For $r > 1$, the canonical injection $G'^r \rightarrow G^r$ is an isomorphism.*

We give an independent proof of Theorem 4.3.1 without the reduction to the classical case and construct the injection (4.20) for $r > 1$. Let ord_K also denote the extension to a separable closure \bar{K} of the normalized discrete valuation of K and, for a rational number r , set $\mathfrak{m}_{\bar{K}}^r = \{x \in \bar{K} \mid \text{ord}_K x \geq r\} \supset \mathfrak{m}_{\bar{K}}^{r+} = \{x \in \bar{K} \mid \text{ord}_K x > r\}$ so that the \bar{F} -vector space $\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}$ in (4.20) is of dimension 1.

We sketch the proof of Corollary 4.2.6 (1). Using the property that the ramification groups are compatible with quotient [18, Corollary 1.4.3], we reduce the proof to the case where $G^{r+} = 1$ and $\mathrm{Gr}^r G = G^r$. The geometric construction of ramification groups recalled in Section 3.3 provides a connected G^r -torsor $X^{(r)\circ}$ over the dual space $\Theta^{(r)}$ of the \bar{F} -vector space on the target of (4.20). Let $k \rightarrow k'$ be a morphism of algebraically closed fields of characteristic $p > 0$ and let E and E' be a k -vector space and a k' -vector space. For a k' -linear morphism $E' \rightarrow E \otimes_k k'$, the morphism $\pi_1(E', 0)_{\mathrm{pro-}p} \rightarrow \pi_1(E, 0)_{\mathrm{pro-}p}$ of the pro- p quotients of the étale fundamental groups is surjective if and only if the morphism $E' \rightarrow E$ of schemes is dominant by Corollary 2.1.2.1. Using this property and the fact that G^r is a p -group, we prove Corollary 4.2.6 (1).

We sketch the second proof of Theorem 4.3.1 and the construction of (4.20). In Section 2, we prove some properties Proposition 2.2.3 on finite étale isogenies of smooth connected group schemes and the classification Proposition 2.1.1 of extensions by \mathbf{F}_p of a vector space over a field of characteristic $p > 0$, by the dual space. Then, it is reduced to showing that the G^r -torsor $X^{(r)\circ}$ admits a group structure such that the finite étale morphism $X^{(r)\circ} \rightarrow \Theta^{(r)}$ is a morphism of group schemes over \bar{F} . The construction of a group structure on $X^{(r)\circ}$ satisfying the required property is reduced by Proposition 2.2.4 to a construction of an automorphism of the base change $X_{\bar{F}'}^{(r)\circ}$ to an extension \bar{F}' of the function field of $\Theta^{(r)}$ compatible with the translation of $\Theta_{\bar{F}'}^{(r)}$ by the generic point.

By using Néron's desingularization, we reduce the construction to the case where K is essentially of finite type and is separable over a discrete valuation field K_0 with perfect residue field. By constructing a suitable ramified extension of K , we further reduce it to the case where the index $r \geq 2$ is an integer.

In the case where $r \geq 2$ is an integer, an automorphism of the base change $X_{\bar{F}'}^{(r)\circ}$ satisfying the required property is constructed in Section 3.4 as follows. First, we define a deformation L'_1 of the composition field $L_1 = LK_1$ over the local field K_1 at a point of a smooth scheme over \mathcal{O}_K appearing in the construction of the G^r -torsor $X^{(r)\circ}$. By the construction of L'_1 , the étale morphism $X_1^{(r)\circ} \rightarrow \Theta_1^{(r)}$ for the extension L'_1 of K_1 is given by the generic translation of the scalar extension to \bar{F}' of $X^{(r)\circ} \rightarrow \Theta^{(r)}$ for the original extension L over K . We prove that over an unramified extension K_2 of K_1 , the composition field $L'_2 = L'_1 K_2$ is isomorphic to LK_2 , using the defining property of ramification groups and a descent property of étaleness of finite covering with a partial section proved in Section 3.2. This isomorphism induces a required isomorphism from $X_1^{(r)\circ} \rightarrow \Theta_1^{(r)}$ to $X_{\bar{F}'}^{(r)\circ} \rightarrow \Theta_{\bar{F}'}^{(r)}$.

In the geometric case where \mathcal{O}_K is the henselization of a local ring $\mathcal{O}_{X,\xi}$ at the generic point ξ of a smooth divisor D of a smooth scheme X over a perfect field k , such a group structure is deduced in [17] from the groupoid structure on $\mathrm{pr}_1, \mathrm{pr}_2: X \times X \rightarrow X$ defined by $\mathrm{pr}_{13}: (X \times X) \times_X (X \times X) = X \times X \times X \rightarrow X \times X$ and its modification along the boundary obtained by taking blow-up along $R = rD$ embedded in the diagonal $X \subset X \times X$. The

construction in this article is an approximation of the construction in the geometric case.

The contents of each section are briefly described at the beginning of each section.

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1. TANGENT SPACES, IMMERSIONS AND DIFFERENTIALS

First, we define a tangent space at a geometric point of a scheme as the dual of H_1 of the cotangent complex in Section 1.1. Using the tangent spaces, we define a condition for a morphism of schemes to be tangentially dominant at a point. This is weaker than formal smoothness and will be used to construct an extension of discrete valuation field with perfect field in the first proof of Theorem 4.3.1.

In Section 1.2, we study closed immersions of the spectrum of a finite extension of discrete valuation rings to smooth schemes, used in the geometric definition of the filtration by ramification groups in Section 3.3. We study

the case where the discrete valuation ring is essentially of finite type over a discrete valuation ring with perfect residue field, which will be used in a reduction step to the case where the index $r > 1$ is an integer of the second proof of Theorem 4.3.1.

In Section 1.3, we study the relation of the conormal sheaf of closed immersion with the tangent space introduced in Section 1.1 and with the module of differentials. The relation with the tangent space will be used in the reduction to the perfect residue field case and also to show the triviality of a Galois action in Section 3.3. The relation with the module of differentials is used in the reduction step to the case where the index $r > 1$ is an integer of the second proof.

1.1. Tangent space at a point of a scheme. We briefly recall basic facts on cotangent complexes from [12, Chapitres II, III]. For a morphism of schemes $X \rightarrow S$, the cotangent complex $L_{X/S}$ is defined [12, Chapitre II, 1.2.3] as a chain complex of flat \mathcal{O}_X -modules, whose cohomology sheaves are quasi-coherent. If $X = \text{Spec } A$ and $S = \text{Spec } B$ are affine, there exist a complex $L_{A/B}$ of flat A -modules and a canonical quasi-isomorphism $L_{A/B} \otimes_A \mathcal{O}_X \rightarrow L_{X/S}$.

The cohomology sheaf $\mathcal{H}_1(L_{X/S})$ is closely related to the module of imperfection studied in [10, Chapitre 0, Section 20.6]. Later Grothendieck introduced the truncation $\tau_{[-1,0]}L_{X/S}$ denoted $L^{X/S}$ in his prenotes for Hartshorne's seminar on Residues and Duality and called $\mathcal{H}_1(L^{X/S})$ the conormal module of X relatively to S . The author thanks Luc Illusie for telling him the history.

In this section, we study the case where $X = \text{Spec } E$ for a field E of characteristic $p > 0$ and show that the E -vector space $H_1(L_{E/S})$ plays a role of the cotangent space of S at the point defined by $\text{Spec } E \rightarrow S$. We study a globalization in [21, Section 4].

There is a canonical isomorphism $\mathcal{H}_0(L_{X/S}) \rightarrow \Omega_{X/S}^1$ [12, Chapitre II, Proposition 1.2.4.2]. This induces a canonical morphism $L_{X/S} \rightarrow \Omega_{X/S}^1[0]$. For a commutative diagram

$$(1.1) \quad \begin{array}{ccc} X' & \longrightarrow & S' \\ f \downarrow & & \downarrow \\ X & \longrightarrow & S, \end{array}$$

a canonical morphism $Lf^*L_{X/S} \rightarrow L_{X'/S'}$ is defined [12, Chapitre II, (1.2.3.2)']. For a morphism $f: X \rightarrow Y$ of schemes over a scheme S , a distinguished triangle

$$(1.2) \quad Lf^*L_{Y/S} \rightarrow L_{X/S} \rightarrow L_{X/Y} \rightarrow$$

is defined [12, Chapitre II, Proposition 2.1.2].

If $X \rightarrow S$ is a closed immersion defined by the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_S$ and if $N_{X/S} = \mathcal{I}_X/\mathcal{I}_X^2$ denotes the conormal sheaf, there exists a canonical isomorphism $\mathcal{H}_1(L_{X/S}) \rightarrow N_{X/S}$ [12, Chapitre III, Corollaire 1.2.8.1]. This induces a canonical morphism $L_{X/S} \rightarrow N_{X/S}[1]$.

Lemma 1.1.1. 1. ([12, Chapitre III, Proposition 1.2.9]) *Let $f: X \rightarrow Y$ be an immersion of schemes over a scheme S . Then, the boundary morphism $\partial: N_{X/Y} \rightarrow f^*\Omega_{Y/S}^1$ of the distinguished triangle $Lf^*L_{Y/S} \rightarrow L_{X/S} \rightarrow L_{X/Y} \rightarrow$ sends g to $-dg$.*

2. ([12, Chapitre III, Proposition 3.1.2 (i) \Rightarrow (ii)]) *Let $X \rightarrow S$ be a smooth morphism. Then, the canonical morphism $L_{X/S} \rightarrow \Omega_{X/S}^1[0]$ is a quasi-isomorphism.*

3. ([12, Chapitre III, Proposition 3.2.4 (iii)]) *If $X \rightarrow S$ is a regular immersion, the canonical morphism $L_{X/S} \rightarrow N_{X/S}[1]$ is a quasi-isomorphism.*

4. ([11, 9.5.8] cf. [12, Chapitre III, Remarque 3.1.4]) *Let $f: X \rightarrow S$ be a formally smooth morphism. Then, $\Omega_{X/S}^1$ is a flat \mathcal{O}_X -module and $H_1(L_{X/S}) = 0$.*

Lemma 1.1.2. *Let $F \rightarrow E$ be an extension of fields. Then, the cotangent complex $L_{E/F}$ is acyclic except at degree $[-1, 0]$. If E is a separable extension of F , the canonical morphism $L_{E/F} \rightarrow \Omega_{E/F}^1[0]$ is a quasi-isomorphism. If F is of characteristic $p > 0$ and if $k \subset F^p$ is a subfield, the minus of the boundary morphism of the distinguished triangle $L_{F/k} \otimes_F E \rightarrow L_{E/k} \rightarrow L_{E/F} \rightarrow$ defines an isomorphism*

$$(1.3) \quad -\partial: H_1(L_{E/F}) \rightarrow \text{Ker}(\Omega_{F/k}^1 \otimes_F E \rightarrow \Omega_{E/k}^1).$$

Proof. First, we show the case where E is a separable extension of F . By taking the limit [12, (1.2.3.4)], we may assume that E is finitely generated over F . We may further assume that E is the function field of a smooth scheme X over F . Then, we have a quasi-isomorphism $L_{X/F} \rightarrow \Omega_{X/F}^1[0]$ by Lemma 1.1.1.2. Hence $L_{E/F} \rightarrow \Omega_{E/F}^1[0]$ is a quasi-isomorphism and $\Omega_{F/k}^1 \otimes_F E \rightarrow \Omega_{E/k}^1$ is an injection in this case.

To show the remaining assertion, we may assume that F is of characteristic $p > 0$. Let $k_0 \subset F$ be a perfect subfield, say $k_0 = \mathbf{F}_p$. Then, since F and E are separable extensions of k_0 , the distinguished triangle $L_{F/k_0} \otimes_F E \rightarrow L_{E/k_0} \rightarrow L_{E/F} \rightarrow$ implies that $L_{E/F}$ is acyclic except at degree $[-1, 0]$ and defines an isomorphism $H_1(L_{E/F}) \rightarrow \text{Ker}(\Omega_{F/k_0}^1 \otimes_F E \rightarrow \Omega_{E/k_0}^1)$ (1.3) for $k = k_0$. If $k \subset F^p$ is a subfield, $\Omega_{F/k}^1 \rightarrow \Omega_{F/F^p}^1$ and $\Omega_{E/k}^1 \rightarrow \Omega_{E/F^p}^1$ are isomorphisms and the right hand side of (1.3) is independent of such k , including k_0 . \square

Proposition 1.1.3. *Let S be a scheme, $s \in S$ be a point and E be an extension of the residue field $F = k(s)$ at s .*

1. *The distinguished triangle*

$$(1.4) \quad L_{F/S} \otimes_F E \rightarrow L_{E/S} \rightarrow L_{E/F} \rightarrow$$

of cotangent complexes and the isomorphism (1.3) define an isomorphism $H_0(L_{E/S}) \rightarrow \Omega_{E/F}^1$ and an exact sequence

$$(1.5) \quad 0 \rightarrow \mathfrak{m}_s/\mathfrak{m}_s^2 \otimes_F E \rightarrow H_1(L_{E/S}) \xrightarrow{-\partial} \text{Ker}(\Omega_F^1 \otimes_F E \rightarrow \Omega_E^1) \rightarrow 0$$

of E -vector spaces.

2. Assume that F is of characteristic $p > 0$ and that E is an extension of $F^{1/p}$. Then, the exact sequence (1.5) defines an exact sequence

$$(1.6) \quad 0 \rightarrow \mathfrak{m}_s/\mathfrak{m}_s^2 \otimes_F E \rightarrow H_1(L_{E/S}) \xrightarrow{-\partial} \Omega_F^1 \otimes_F E \rightarrow 0.$$

3. Let $E \subset E'$ be extensions of F . Then, the canonical morphism

$$(1.7) \quad H_1(L_{E/S}) \otimes_E E' \rightarrow H_1(L_{E'/S})$$

is an injection. This is an isomorphism either if F is of characteristic 0 or if F is of characteristic $p > 0$ and E is an extension of $F^{1/p}$.

4. If S is regular at s , then the cotangent complex $L_{E/S}$ is acyclic except at degree $[-1, 0]$. Further if $\Omega_{E/F}^1 = 0$, the complex $L_{E/S}$ is acyclic except at degree -1 .

The exact sequence (1.6) in the case $E = F^{1/p}$ is canonically identified with that constructed in [8, Proposition 9.6.14] by a different method ([21, Corollary 2.7, Corollary 4.12]). The converse of Proposition 1.1.3.4 holds by [8, Corollary 9.6.45]. The exact sequence (1.5) implies that $H_1(L_{E/S})$ is determined by $\mathcal{O}_{S,s}/\mathfrak{m}_s^2 \rightarrow E$.

Proof. 1. We have $H_1(L_{F/S}) = \mathfrak{m}_s/\mathfrak{m}_s^2$ and $H_0(L_{F/S}) = 0$. By Lemma 1.1.2, we have $H_0(L_{E/F}) = \Omega_{E/F}^1$, (1.3) and $H_2(L_{E/F}) = 0$. Hence the distinguished triangle (1.4) defines an isomorphism $H_0(L_{E/S}) \rightarrow \Omega_{E/F}^1$ and the exact sequence (1.5).

2. Assume $F^{1/p} \subset E$. Then, the morphism $\Omega_F^1 \otimes_F E \rightarrow \Omega_E^1$ is 0 and we have $\text{Ker}(\Omega_F^1 \otimes_F E \rightarrow \Omega_E^1) = \Omega_F^1 \otimes_F E$. Hence (1.5) gives (1.6).

3. Since $L_{E'/E}$ is acyclic except at degree $[-1, 0]$ by Lemma 1.1.2, the distinguished triangle $L_{E/S} \otimes_E E' \rightarrow L_{E'/S} \rightarrow L_{E'/E} \rightarrow$ implies the injectivity of (1.7).

If F is of characteristic 0, we have $H_1(L_{E'/E}) = 0$ and (1.7) is a surjection. If $F^{1/p} \subset E$, the boundary mapping $H_1(L_{E'/E}) \rightarrow \Omega_{E/F}^1 \otimes_E E'$ is an injection by Lemma 1.1.2. Hence (1.7) is a surjection.

4. The morphism $L_{F/S} \rightarrow N_{F/S}[1]$ is a quasi-isomorphism by Lemma 1.1.1.3. Hence by Lemma 1.1.2, the distinguished triangle (1.4) shows that $L_{E/S}$ is acyclic except at degree $[-1, 0]$. Further if $H_0(L_{E/S}) = \Omega_{E/F}^1$ is 0, then $L_{E/S}$ is acyclic except at degree -1 . \square

For a morphism $S' \rightarrow S$ of schemes sending a point $s' \in S'$ to $s \in S$ and a morphism $E \rightarrow E'$ of extensions extending the morphism $F = k(s) \rightarrow F' = k(s')$ of the residue fields, by the functoriality of cotangent complexes, we have a canonical morphism

$$(1.8) \quad H_1(L_{E/S}) \rightarrow H_1(L_{E'/S'}).$$

Let S be a scheme, $s \in S$ be a point and $u \in \mathcal{O}_{S,s}$. Assume that the residue field $F = k(s)$ is of characteristic $p > 0$ and let E be an extension of F containing a p -th root $v \in E$ of $\bar{u} \in F$. Then,

$$(1.9) \quad \tilde{d}u \in H_1(L_{E/S})$$

is defined as $w(u, v)$ in [21, Definition 4.6]. We briefly recall the construction. Let $W \subset \mathbf{A}_S^1 = S \times_{\text{Spec } \mathbf{Z}} \text{Spec } \mathbf{Z}[T]$ be the closed subscheme defined on a neighborhood of s by $T^p - u$ and define a morphism $\text{Spec } E \rightarrow W$ over S by sending T to v . Then, the image of $\tilde{d}u$ by the injection $H_1(L_{E/S}) \rightarrow H_1(L_{E/\mathbf{A}_S^1})$ is the image of the section $u - T^p$ of the conormal sheaf N_{W/\mathbf{A}_S^1} . The construction of $\tilde{d}u$ is functorial. The following property is a special case of [21, Proposition 4.7].

Lemma 1.1.4. *Let S be a scheme, $s \in S$ be a point and $u \in \mathcal{O}_{S,s}$. Assume that the residue field $F = k(s)$ is of characteristic $p > 0$ and let E be an extension of F containing a p -th root v of $\bar{u} \in F$.*

1. (cf. [8, 9.6.12]) *Let $u, u' \in \mathcal{O}_{S,s}$ such that there exist p -th roots $v, v' \in E$ of \bar{u}, \bar{u}' . Then, we have*

$$(1.10) \quad \tilde{d}(u + u') = \tilde{d}u + \tilde{d}u' + \sum_{i=1}^{p-1} \frac{v^i v'^{(p-i)}}{i! (p-i)!} \cdot \tilde{d}p,$$

$$(1.11) \quad \tilde{d}(uu') = u' \cdot \tilde{d}u + u \cdot \tilde{d}u'.$$

2. *The image of $\tilde{d}u \in H_1(L_{E/S})$ (1.9) by $-\partial: H_1(L_{E/S}) \rightarrow \Omega_F^1 \otimes_F E$ (1.5) is $d\bar{u}$.*

3. *If $u \equiv 0 \pmod{\mathfrak{m}_s}$, we have $\omega \equiv u \in \mathfrak{m}_s/\mathfrak{m}_s^2 = H_1(L_{F/S}) = H_1(L_{E/S})$.*

By (1.10), if one of u, u' is an element of the maximal ideal $\mathfrak{m}_s \subset \mathcal{O}_{S,s}$, we have $\tilde{d}(u + u') = \tilde{d}u + \tilde{d}u'$. By (1.11), we have $\tilde{d}u^i = iu^{i-1}\tilde{d}u$ by induction on integer $i \geq 0$. Hence for an integer a , since $a - a^p$ is divisible by p , we have

$$\tilde{d}a = \tilde{d}a^p + \tilde{d}(a - a^p) = pa^{p-1} \cdot \tilde{d}a + p \cdot \tilde{d} \frac{a - a^p}{p} + \frac{a - a^p}{p} \cdot \tilde{d}p = \frac{a - a^p}{p} \cdot \tilde{d}p.$$

In the case where $v \notin F = k(s)$, the following remark is due to Luc Illusie. Let $S' = \text{Spec } \mathcal{O}_{S,s}[T]/(T^p - u) \rightarrow S$ and identify $s' = \text{Spec } F(v)$ with the closed point of S' . Then, the morphism $H_1(L_{F(v)/S}) \rightarrow H_1(L_{F(v)/S'})$ defines a splitting of the exact sequence $0 \rightarrow H_1(L_{F/S}) \otimes_F F(v) \rightarrow H_1(L_{F(v)/S}) \rightarrow H_1(L_{F(v)/F}) \rightarrow 0$ defined by the distinguished triangle $L_{F/S} \otimes_F^L F(v) \rightarrow L_{F(v)/S} \rightarrow L_{F(v)/F} \rightarrow$ and the image of $\tilde{d}u \in H_1(L_{F(v)/S})$ in $H_1(L_{F(v)/S'})$ is $\tilde{d}T^p = pT^{p-1}\tilde{d}T = 0$ by the Leibniz rule.

Proposition 1.1.5. *Let $S \rightarrow S_0$ be a morphism of schemes, let $s \in S$ be a point and $s_0 \in S_0$ be its image. Let $F = k(s)$ be the residue field and assume that $k = k(s_0)$ is a perfect field of characteristic $p > 0$. Let E be an extension of F and define a morphism*

$$(1.12) \quad -\partial: H_1(L_{E/S}) \rightarrow \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} E.$$

to be the minus of the boundary morphism of the distinguished triangle $L_{S/S_0} \otimes_{\mathcal{O}_K}^L E \rightarrow L_{E/S_0} \rightarrow L_{E/S} \rightarrow$ defined by $\text{Spec } E \rightarrow S \rightarrow S_0$.

1. (cf. [8, Lemma 9.6.3]) *For $E = F$, the above distinguished triangle defines an exact sequence*

$$(1.13) \quad 0 \rightarrow \mathfrak{m}_s/(\mathfrak{m}_s^2 + \mathfrak{m}_{s_0}\mathcal{O}_{S,s}) \xrightarrow{d} \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} F \rightarrow \Omega_{F/k}^1 \rightarrow 0$$

of F -vector spaces.

2. ([21, Proposition 4.7.3]) For $u \in \mathcal{O}_{S,s}$, the mapping (1.12) sends $\tilde{d}u$ to du .

3. The mapping (1.12) fits in the commutative diagram

$$(1.14) \quad \begin{array}{ccc} H_1(L_{E/S}) & \xrightarrow{(1.5)} & \Omega_{F/k}^1 \otimes_F E \\ (1.5) \uparrow & \searrow (1.12) & \uparrow \\ \mathfrak{m}_s/\mathfrak{m}_s^2 \otimes_F E & \xrightarrow{d} & \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} E \end{array}$$

of E -vector spaces. The kernel and the cokernel of (1.12) are the images of $\mathfrak{m}_{s_0}/\mathfrak{m}_{s_0}^2 \otimes_F E \rightarrow \mathfrak{m}_s/\mathfrak{m}_s^2 \otimes_F E$ and of $\Omega_{F/k}^1 \otimes_F E \rightarrow \Omega_{E/k}^1$. Consequently, the morphism (1.12) is a surjection (resp. an injection) if F is a subfield of E^p (resp. if $\mathfrak{m}_{s_0}\mathcal{O}_{S,s} \subset \mathfrak{m}_s^2$).

Proof. 1. By Lemma 1.1.1.1, the minus of the boundary morphism of the distinguished triangle $L_{S/S_0} \otimes_{\mathcal{O}_S}^L F \rightarrow L_{F/S_0} \rightarrow L_{F/S} \rightarrow$ defines an exact sequence

$$(1.15) \quad H_1(L_{F/S_0}) \rightarrow \mathfrak{m}_s/\mathfrak{m}_s^2 \xrightarrow{d} \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} F \rightarrow \Omega_{F/k}^1 \rightarrow 0$$

of F -vector spaces. Since $\Omega_k^1 = 0$, the exact sequence (1.5) gives an isomorphism $\mathfrak{m}_{s_0}/\mathfrak{m}_{s_0}^2 \otimes_k F \rightarrow H_1(L_{F/S_0})$ and we obtain (1.13).

3. The commutative diagram

$$\begin{array}{ccccc} L_{S/S_0} \otimes_{\mathcal{O}_S}^L F \otimes_F E & \xlongequal{\quad} & L_{S/S_0} \otimes_{\mathcal{O}_S}^L E & \longrightarrow & L_{F/S_0} \otimes_F E \\ \downarrow & & \downarrow & & \downarrow \\ L_{F/S_0} \otimes_F E & \longrightarrow & L_{E/S_0} & \xlongequal{\quad} & L_{E/S_0} \\ \downarrow & & \downarrow & & \downarrow \\ L_{F/S} \otimes_F E & \longrightarrow & L_{E/S} & \longrightarrow & L_{E/F} \\ \downarrow & & \downarrow & & \downarrow \end{array}$$

of distinguished triangles defines the commutative diagram (1.14) by Lemma 1.1.1.1. Similarly as in the proof of 1, we obtain an exact sequence

$$(1.16) \quad \mathfrak{m}_{s_0}/\mathfrak{m}_{s_0}^2 \otimes_k E \rightarrow H_1(L_{E/S}) \rightarrow \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} E \rightarrow \Omega_{E/k}^1.$$

Since k is perfect, we have $\Omega_{F/k}^1 = \Omega_F^1$. Hence the assertion on the kernel and the cokernel follows from (1.16), $(1.15) \otimes_F E$, (1.5).

If $F^{1/p} \subset E$, then $\Omega_F^1 \otimes_F E \rightarrow \Omega_E^1$ is the 0-mapping. If $\mathfrak{m}_{s_0}\mathcal{O}_{S,s} \subset \mathfrak{m}_s^2$, then $\mathfrak{m}_{s_0}/\mathfrak{m}_{s_0}^2 \rightarrow \mathfrak{m}_s/\mathfrak{m}_s^2$ is the 0-mapping. \square

Definition 1.1.6. Let S be a scheme, $s \in S$ be a point and $F = k(s)$ be the residue field.

1. For an extension E of F , we call the spectrum

$$(1.17) \quad \Theta_{S,E} = \text{Spec } S(H_1(L_{E/S}))$$

of the symmetric algebra over E the tangent space of S at E . If $S = \text{Spec } \mathcal{O}_K$ for a discrete valuation ring \mathcal{O}_K and if $s \in S$ is the closed point, we also write $\Theta_{S,E} = \Theta_{K,E}$.

2. Let $S' \rightarrow S$ be a morphism of schemes and let $s \in S$ be the image of $s' \in S'$. Let $\bar{F} \rightarrow \bar{F}'$ be a morphism of algebraic closures of the residue fields extending $F = k(s) \rightarrow F' = k(s')$. We say that the morphism $S' \rightarrow S$ is dominant on the tangent spaces, or tangentially dominant, at s' if the morphism

$$S(H_1(L_{\bar{F}/S})) \rightarrow S(H_1(L_{\bar{F}'/S'}))$$

is an injection.

3. Let \mathcal{O}_K and $\mathcal{O}_{K'}$ be discrete valuation rings. We say that a morphism $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ is a morphism of discrete valuation rings if it is faithfully flat. We say that a morphism $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ of discrete valuation rings is tangentially dominant if $S' = \text{Spec } \mathcal{O}_{K'} \rightarrow S = \text{Spec } \mathcal{O}_K$ is dominant on the tangent space at the closed point of S' .

For $E = F$, we recover the definition of the Zariski tangent space. That a morphism $S' \rightarrow S$ is dominant on the tangent spaces at s' means that the morphism $\Theta_{S',\bar{F}'} \rightarrow \Theta_{S,\bar{F}}$ of schemes is dominant.

Lemma 1.1.7. *Let S be a scheme, $s \in S$ a point and E be an extension of the residue field $F = k(s)$.*

Let $S' \rightarrow S$ be a morphism of schemes sending $s' \in S'$ to $s \in S$ and let $E \rightarrow E'$ be a morphism of extensions of residue fields. If $S' \rightarrow S$ is formally smooth at s' , the morphism (1.8) induces an injection

$$(1.18) \quad H_1(L_{E/S}) \otimes_E E' \rightarrow H_1(L_{E'/S'})$$

of E' -vector spaces.

Proof. By Proposition 1.1.3.3, we may assume that $E = E'$. By Lemma 1.1.1.4, we have $H_1(L_{S'/S} \otimes_S^L E') = 0$. Hence the distinguished triangle $L_{S'/S} \otimes_S^L E' \rightarrow L_{E'/S} \rightarrow L_{E'/S'}$ implies the injectivity of (1.18). \square

Proposition 1.1.8. 1. *Let $S' \rightarrow S$ be a morphism of schemes and let $s \in S$ be the image of $s' \in S'$. We consider the following conditions:*

- (1) *The morphism $S' \rightarrow S$ is formally smooth at s' .*
- (2) *The morphism $S' \rightarrow S$ is dominant on the tangent spaces at s' .*
- (3) *The morphism $\mathfrak{m}_s/\mathfrak{m}_s^2 \rightarrow \mathfrak{m}_{s'}/\mathfrak{m}_{s'}^2$ is injective.*

Then, we have the implications (1) \Rightarrow (2) \Rightarrow (3).

2. *Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of discrete valuation rings. We consider the following conditions:*

- (1) *The ramification index $e_{K'/K}$ is 1 and $F' = \mathcal{O}_{K'}/\mathfrak{m}_{K'}$ is a separable extension of $F = \mathcal{O}_K/\mathfrak{m}_K$.*
- (2) *The morphism $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ is tangentially dominant.*
- (3) *The ramification index $e_{K'/K}$ is 1.*

Then, we have the implications (1) \Rightarrow (2) \Rightarrow (3).

Proof. 1. (1) \Rightarrow (2): Let $\bar{F} \rightarrow \bar{F}'$ be a morphism of algebraic closures of $F = k(s)$ and $F' = k(s')$. Then, by Lemma 1.1.7.2, the morphism $H_1(L_{\bar{F}/S}) \otimes_{\bar{F}}$

$\bar{F}' \rightarrow H_1(L_{\bar{F}'/S'})$ is an injection. Hence $S(H_1(L_{\bar{F}/S})) \rightarrow S(H_1(L_{\bar{F}'/S})) \otimes_{\bar{F}} \bar{F}' \rightarrow S(H_1(L_{\bar{F}'/S'}))$ is an injection.

(2) \Rightarrow (3): By Proposition 1.1.3, the injection $S(H_1(L_{\bar{F}/S})) \rightarrow S(H_1(L_{\bar{F}'/S'}))$ induces an injection $\mathfrak{m}_s/\mathfrak{m}_s^2 \rightarrow \mathfrak{m}_{s'}/\mathfrak{m}_{s'}^2$.

2. Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of discrete valuation rings. Let $S' = \text{Spec } \mathcal{O}_{K'} \rightarrow S = \text{Spec } \mathcal{O}_K$ be the associated morphism and $s' \in S'$ be the closed point. Then by [10, Chapitre 0, Théorème (19.7.1)], the condition (1) in 1. is equivalent to that $\mathcal{O}_{K'}/\mathfrak{m}_K \mathcal{O}_{K'}$ is formally smooth over the residue field $F = \mathcal{O}_K/\mathfrak{m}_K$. By [10, Chapitre 0, Corollaire (19.6.5)], this implies that $\mathfrak{m}_{K'} = \mathfrak{m}_K \mathcal{O}_{K'}$ and that $F' = \mathcal{O}_{K'}/\mathfrak{m}_{K'}$ is a separable extension of F . Since the converse is true by [10, Chapitre 0, Théorème (19.6.1)], the condition (1) in 1. is equivalent to the condition (1) in 2.

Further, the condition (3) in 1. is equivalent to the condition (3) in 2. \square

The implications (1) \Rightarrow (2) \Rightarrow (3) are strict as follows.

Example 1.1.9. Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of discrete valuation rings. Assume that the residue field F is of characteristic $p > 0$ and let $u \in \mathcal{O}_K$ and $v \in \mathcal{O}_{K'}, w \in \mathfrak{m}_{K'}$ be elements satisfying $u = v^p + w$. Then, by Lemma 1.1.4.1, we have

$$(1.19) \quad \tilde{d}u = pv^{p-1}\tilde{d}v + \tilde{d}w = \tilde{d}w$$

in $H_1(L_{\bar{F}'/S})$. Hence, if $du \neq 0$ in Ω_F^1 and $w \in \mathfrak{m}_{K'}^2$, the morphism $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ is not tangentially dominant.

Concretely, for $K = k(u)((t))$, the separable extension $K' = K(v)$ of degree p defined by $v^p - t^2v = u$ is not tangentially dominant. Hence the condition (3) is satisfied but (2) is not. \square

The following Proposition gives an example of extension satisfying the condition (2) but not (1).

Proposition 1.1.10. *Let K be a discrete valuation field. Then, there exists a tangentially dominant morphism $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ of discrete valuation rings such that the residue field F' is perfect.*

Proof. It suffices to consider the case where the residue field F is of characteristic $p > 0$. Let $t \in \mathcal{O}_K$ be a uniformizer and let $(u_i)_{i \in I}$ be a p -basis of the residue field F . For each $i \in I$, take a lifting of $u_i \in F$ in \mathcal{O}_K and let it also denoted u_i by abuse of notation. The localization \mathcal{O}_{K_0} of the polynomial ring $\mathcal{O}_K[w_{i,0} (i \in I)]$ at the prime ideal (t) is a discrete valuation ring. We set $v_{i,0} = u_i - w_{i,0}t \in \mathcal{O}_{K_0}$ for each $i \in I$ so that $(v_{i,0}, w_{i,0})_{i \in I}$ is a p -basis of the residue field $F_0 = F(w_{i,0} (i \in I))$ of K_0 .

We define an inductive system $(\mathcal{O}_{K_n})_{n \in \mathbf{N}}$ of discrete valuation rings inductively by

$$\mathcal{O}_{K_n} = \mathcal{O}_{K_{n-1}}[v_{i,n}, w_{i,n} (i \in I)] / (v_{i,n}^p - v_{i,n-1}, w_{i,n}^p - w_{i,n-1} (i \in I)).$$

The inductive limit $\mathcal{O}_{K'} = \varinjlim_{n \in \mathbf{N}} \mathcal{O}_{K_n}$ is a discrete valuation ring and the residue field F' of K' is a perfect closure of F_0 and is perfect.

The uniformizer t defines a basis of $\mathfrak{m}_K/\mathfrak{m}_K^2$ and $(du_i)_{i \in I}$ defines a basis of Ω_F^1 . Hence by the exact sequence $0 \rightarrow \mathfrak{m}_K/\mathfrak{m}_K^2 \otimes_F \bar{F} \rightarrow H_1(L_{\bar{F}/S}) \rightarrow$

$\Omega_{\bar{F}}^1 \otimes_{\bar{F}} \bar{F} \rightarrow 0$ (1.6) and Lemma 1.1.4, $\tilde{d}t$ together with $(\tilde{d}u_i)_{i \in I}$ defines a basis of $H_1(L_{\bar{F}/S})$. By (1.19), the images of $\tilde{d}u_i \in H_1(L_{\bar{F}/S})$ in $H_1(L_{\bar{F}'/S'}) = \mathfrak{m}_{K'}/\mathfrak{m}_{K'}^2 \otimes_{\bar{F}'} \bar{F}'$ is $w_{i,0}t$. Since $(w_{i,0})_{i \in I}$ are algebraically independent over \bar{F} , the morphism $S(H_1(L_{\bar{F}/S})) \rightarrow S(H_1(L_{\bar{F}'/S'}))$ is an injection. \square

1.2. Immersions to smooth schemes.

Lemma 1.2.1. *Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings and set $S = \text{Spec } \mathcal{O}_K$ and $T = \text{Spec } \mathcal{O}_L$.*

1. *Let $T \rightarrow Q$ be an immersion to a smooth scheme over S . Then, after replacing Q by a neighborhood of T , there exist a smooth scheme P over S and a cartesian diagram*

$$(1.20) \quad \begin{array}{ccc} Q & \longleftarrow & T \\ \downarrow & & \downarrow \\ P & \longleftarrow & S \end{array}$$

of schemes over S such that $Q \rightarrow P$ is quasi-finite and flat.

2. *Let $T \rightarrow Q$ and $T \rightarrow Q'$ be closed immersions to smooth schemes over S . Then, after replacing Q by a neighborhood of T , there exists a commutative diagram*

$$(1.21) \quad \begin{array}{ccc} T & \longrightarrow & Q' \\ \downarrow & \searrow & \uparrow \\ Q & \longleftarrow & Q'' \end{array}$$

of schemes over S such that $Q'' \rightarrow Q'$ is étale.

3. *Let $T \rightarrow Q$ and $T \rightarrow Q'$ be closed immersions to smooth schemes over S and let*

$$(1.22) \quad \begin{array}{ccc} T & \longrightarrow & Q' \\ \downarrow & \searrow & \\ Q & & \end{array}$$

be a commutative diagram of schemes over S such that $Q' \rightarrow Q$ is smooth. Then, after replacing Q' by a neighborhood of T , there exists a commutative diagram

$$(1.23) \quad \begin{array}{ccc} T & \longrightarrow & Q' \\ \downarrow & \searrow & \downarrow \\ Q & \longleftarrow & \mathbf{A}_Q^n \end{array}$$

of schemes over S such that $Q' \rightarrow \mathbf{A}_Q^n$ is étale, $\mathbf{A}_Q^n \rightarrow Q$ is the projection and the composition $T \rightarrow \mathbf{A}_Q^n$ is the composition of $T \rightarrow Q$ with the 0-section $Q \rightarrow \mathbf{A}_Q^n$.

Proof. 1. Since T and Q are regular, the immersion $T \rightarrow Q$ is a regular immersion by [10, Chap. 0 Corollaire (17.1.9) a) \Rightarrow b)]. After replacing Q

by a neighborhood of T , we may assume that the closed subscheme $T \subset Q$ is defined by a regular sequence t_1, \dots, t_n . We define $Q \rightarrow P = \mathbf{A}_S^n$ by t_1, \dots, t_n and $S \rightarrow P$ to be the 0-section. Then, we obtain a cartesian diagram (1.20). Since $T \rightarrow S$ is flat and t_1, \dots, t_n is a regular sequence, after further replacing Q by a neighborhood of T , the morphism $Q \rightarrow P$ is flat and quasi-finite by [10, Chapitre 0, Proposition (15.1.21) b) \Rightarrow a)].

2. Since Q is smooth over S , there exists a commutative diagram (1.21) such that $Q' \rightarrow Q$ is étale.

3. Since the commutative diagram (1.22) defines a section $T \rightarrow Q' \times_Q T$ of a smooth morphism $Q' \times_Q T \rightarrow T$, locally on a neighborhood of $T \subset Q' \times_Q T$, there exist an integer $n \geq 0$ and an étale morphism $Q' \times_Q T \rightarrow \mathbf{A}_T^n$ such that the composition $T \rightarrow Q' \times_Q T \rightarrow \mathbf{A}_T^n$ is the 0-section. Hence, it suffices to take a lifting $Q' \rightarrow \mathbf{A}_Q^n$ on a neighborhood of T . \square

Definition 1.2.2. Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings and E be the residue field of L . We say that an immersion $T = \text{Spec } \mathcal{O}_L \rightarrow Q$ to a smooth scheme over $S = \text{Spec } \mathcal{O}_K$ is minimal if the canonical surjection $\Omega_{Q/S}^1 \otimes_{\mathcal{O}_Q} E \rightarrow \Omega_{T/S}^1 \otimes_{\mathcal{O}_T} E$ is an isomorphism.

Lemma 1.2.3. Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings and set $S = \text{Spec } \mathcal{O}_K$ and $T = \text{Spec } \mathcal{O}_L$.

1. There exists a minimal immersion $T \rightarrow Q$ to a smooth scheme Q over S .

2. Let $T \rightarrow Q$ and $T \rightarrow Q'$ be closed immersions to smooth schemes over S and let

$$(1.24) \quad \begin{array}{ccc} T & \longrightarrow & Q' \\ & \searrow & \downarrow \\ & & Q \end{array}$$

be a commutative diagram of schemes over S . If $T \rightarrow Q$ is minimal, then $Q' \rightarrow Q$ is smooth on a neighborhood of T .

Proof. 1. Let $a_1, \dots, a_n \in \mathcal{O}_L$ be elements such that da_1, \dots, da_n define a basis of $\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1 \otimes_{\mathcal{O}_L} E$. Then the morphism $T \rightarrow \mathbf{A}_S^n = \text{Spec } \mathcal{O}_K[X_1, \dots, X_n]$ defined by $a_1, \dots, a_n \in \mathcal{O}_L$ is unramified. By [10, Chapitre 4, Corollaire (18.4.7)], there exist an étale scheme Q over \mathbf{A}_S^n and an immersion $T \rightarrow Q$ lifting $T \rightarrow \mathbf{A}_S^n$. Then, $\Omega_{Q/S}^1 \otimes_{\mathcal{O}_Q} E \rightarrow \Omega_{T/S}^1 \otimes_{\mathcal{O}_T} E$ sends a basis dX_1, \dots, dX_n to da_1, \dots, da_n .

2. Since $T \rightarrow Q$ is minimal and $\Omega_{Q'/S}^1 \otimes_{\mathcal{O}_Q} E \rightarrow \Omega_{T/S}^1 \otimes_{\mathcal{O}_T} E$ is a surjection, the morphism $\Omega_{Q'/S}^1 \otimes_{\mathcal{O}_Q} E \rightarrow \Omega_{Q'/S}^1 \otimes_{\mathcal{O}_{Q'}} E$ is an injection of E -vector spaces. Hence, $Q' \rightarrow Q$ is smooth on a neighborhood of T . \square

We will use in Section 3.4 the following description of the fiber of the conormal sheaf.

Lemma 1.2.4. Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings and $T = \text{Spec } \mathcal{O}_L \rightarrow Q$ be a closed immersion to a smooth scheme over $S = \text{Spec } \mathcal{O}_K$. Let $\mathcal{I}_T \subset \mathcal{O}_Q$ be the ideal sheaf defining the closed subscheme

$T \subset Q$ and let $\mathfrak{m}_{Q,t} \subset \mathcal{O}_{Q,t}$ and $\mathfrak{m}_{T,t} \subset \mathcal{O}_{T,t}$ be the maximal ideals at the closed point $t \in T \subset Q$.

1. The inclusion $\mathcal{I}_{T,t} \rightarrow \mathfrak{m}_{Q,t}$ of ideals of $\mathcal{O}_{Q,t}$ induces an isomorphism

$$(1.25) \quad N_{T/Q} \otimes_{\mathcal{O}_T} E \rightarrow \text{Ker}(\mathfrak{m}_{Q,t}/\mathfrak{m}_{Q,t}^2 \rightarrow \mathfrak{m}_{T,t}/\mathfrak{m}_{T,t}^2).$$

2. Let $D \subset Q$ be a regular divisor meeting T transversally at t . Then the isomorphism (1.25) induces an isomorphism

$$(1.26) \quad N_{T/Q} \otimes_{\mathcal{O}_T} E \rightarrow N_{t/D}.$$

Proof. 1. Since the ideal $\mathcal{I}_{T,t} \subset \mathcal{O}_{Q,t}$ of a regular local ring is generated by a part of a system of local parameters, we have an exact sequence

$$0 \rightarrow N_{T/Q} \otimes_{\mathcal{O}_T} E \rightarrow \mathfrak{m}_{Q,t}/\mathfrak{m}_{Q,t}^2 \rightarrow \mathfrak{m}_{T,t}/\mathfrak{m}_{T,t}^2 \rightarrow 0$$

and hence the isomorphism (1.25).

2. By the assumption that D meets T transversally, the product $\mathfrak{m}_{Q,t}/\mathfrak{m}_{Q,t}^2 \rightarrow \mathfrak{m}_{T,t}/\mathfrak{m}_{T,t}^2 \times \mathfrak{m}_{D,t}/\mathfrak{m}_{D,t}^2$ of the canonical surjections is an isomorphism. Since $N_{t/D} = \mathfrak{m}_{D,t}/\mathfrak{m}_{D,t}^2$, the isomorphism (1.25) induces an isomorphism (1.26). \square

Definition 1.2.5. Let $\mathcal{O}_{K_0} \rightarrow \mathcal{O}_L$ be a morphism of discrete valuation rings.

1. We say that \mathcal{O}_L is essentially of finite type (resp. essentially smooth) over \mathcal{O}_{K_0} if there exist a scheme X of finite type (resp. a smooth scheme X) over \mathcal{O}_{K_0} , a point $\xi \in X$ and a morphism $\mathcal{O}_{X,\xi} \rightarrow \mathcal{O}_L$ of discrete valuation rings over \mathcal{O}_{K_0} inducing an isomorphism on the henselizations.
2. Let \mathcal{O}_L be a discrete valuation ring essentially of finite type over \mathcal{O}_{K_0} and let $T = \text{Spec } \mathcal{O}_L \rightarrow Q_0$ be a morphism to a scheme smooth over $S_0 = \text{Spec } \mathcal{O}_{K_0}$. We say that $T \rightarrow Q_0$ is essentially an immersion if the following condition is satisfied: There exist a closed subscheme $X \subset Q_0$, a point $\xi \in X$ and a morphism $\mathcal{O}_{X,\xi} \rightarrow \mathcal{O}_L$ of discrete valuation rings inducing $T \rightarrow Q_0$ and an isomorphism on the henselizations.

We say that an essential immersion $T \rightarrow Q_0$ is divisorial if $X \subset Q_0$ above is a Cartier divisor.

3. Let

$$(1.27) \quad \begin{array}{ccccc} \text{Spec } \mathcal{O}_{K'_0} = S'_0 & \longleftarrow & Q'_0 & \longleftarrow & T' = \text{Spec } \mathcal{O}_{L'} \\ & & \downarrow & & \downarrow \\ & & S_0 & \longleftarrow & Q_0 & \longleftarrow & T \end{array}$$

be a commutative diagram of schemes satisfying the following conditions: The large rectangle is defined by morphisms of discrete valuation rings and the compositions of horizontal arrows are essentially of finite type. The left horizontal arrows are smooth and the right horizontal arrows are essentially immersions.

Let $X \subset Q_0$ and $X' \subset Q'_0$ be as in 2. Then, we say that the right square is essentially cartesian, if the diagram

$$(1.28) \quad \begin{array}{ccc} Q'_0 & \longleftarrow & X' \\ \downarrow & & \downarrow \\ Q_0 & \longleftarrow & X \end{array}$$

is cartesian on a neighborhood of the image of T' .

In Definition 1.2.5.1, we may take X to be regular.

Lemma 1.2.6. *Let $\mathcal{O}_{K_0} \rightarrow \mathcal{O}_K$ be a morphism of discrete valuation rings essentially of finite type. Set $S = \text{Spec } \mathcal{O}_K$ and $S_0 = \text{Spec } \mathcal{O}_{K_0}$.*

1. *Let X be a scheme of finite type, $\xi \in X$ be a point and $S \rightarrow X$ be a morphism over S_0 such that $\mathcal{O}_{X,\xi} \rightarrow \mathcal{O}_K$ induces an isomorphism on henselizations as in Definition 1.2.5.1. Then, X is regular on a neighborhood of ξ .*

2. *Let $S \rightarrow Q_0$ be essentially an immersion to a smooth scheme over S_0 . Let $X \subset Q_0$ be a closed subscheme such that $S \rightarrow Q_0$ induces a morphism $S \rightarrow X$ inducing an isomorphism on the henselizations. Then, the morphism $L_{X/Q_0} \otimes_{\mathcal{O}_X} \mathcal{O}_S \rightarrow L_{S/Q_0}$ induces a quasi-isomorphism $L_{S/Q_0} \rightarrow N_{X/Q_0} \otimes_{\mathcal{O}_X} \mathcal{O}_S[1]$.*

3. *The cotangent complex L_{S/S_0} is of tor-amplitude $[-1, 0]$.*

4. *If K is a separable extension of K_0 , the canonical morphism $L_{S/S_0} \rightarrow \Omega_{S/S_0}^1[0]$ is a quasi-isomorphism.*

In the following, we write $N_{S/Q_0} = H_1(L_{S/Q_0})$. This is canonically isomorphic to the free \mathcal{O}_K -module $N_{X/Q_0} \otimes_{\mathcal{O}_X} \mathcal{O}_K$ of finite rank by Lemma 1.2.6.2.

Proof. 1. Since $S \rightarrow S_0$ is flat and since $\mathfrak{m}_K \subset \mathcal{O}_K$ is generated by a non-zero divisor, we may assume that X is flat over S_0 and that ξ is the generic point of an integral effective Cartier divisor $D \subset X$. Further, we may assume that the generic fiber $X_{K_0} = X \times_S \text{Spec } K_0$ is $X - D$ and is regular. Since D is of finite type over the residue field k of \mathcal{O}_{K_0} , we may also assume that D is regular. Hence X is regular.

2. By 1, we may assume that $X \subset Q_0$ is regular. Hence the immersion $X \rightarrow Q_0$ is a regular immersion and the canonical morphism $L_{X/Q_0} \rightarrow N_{X/Q_0}[1]$ is a quasi-isomorphism by Lemma 1.1.1.3. Since the formation of cotangent complexes commutes with étale morphisms and limits, we obtain a quasi-isomorphism $L_{S/Q_0} \rightarrow N_{X/Q_0} \otimes_{\mathcal{O}_X} \mathcal{O}_S[1]$.

3. By 1, there exists an affine regular scheme $\tilde{X} \subset Q_0 = \mathbf{A}_{S_0}^n$ of finite type over S_0 , a morphism $S \rightarrow \tilde{X}$ over S_0 and $\xi \in \tilde{X}$ be as in Definition 1.2.5.1. Since the formation of cotangent complexes commutes with étale morphisms and limits, we may assume that $\mathcal{O}_K = \mathcal{O}_{\tilde{X},\xi}$. The canonical morphisms $L_{Q_0/S_0} \rightarrow \Omega_{Q_0/S_0}^1[0]$ and $L_{\tilde{X}/Q_0} \rightarrow N_{\tilde{X}/Q_0}[1]$ are quasi-isomorphisms by Lemma 1.1.1.2 and 3. Hence by the distinguished triangle $L_{Q_0/S_0} \otimes_{\mathcal{O}_{Q_0}} \mathcal{O}_X \rightarrow L_{X/S_0} \rightarrow L_{\tilde{X}/Q_0} \rightarrow$, the cotangent complex L_{X/S_0} is of tor-amplitude $[-1, 0]$.

4. Let the notation be as in the proof of 3. If K is separable over K_0 , we may assume that the generic fiber $X_{K_0} = X \times_{S_0} \text{Spec } K_0$ is smooth over K_0 . Since $L_{X_K/K_0} \rightarrow \Omega_{X_K/K_0}^1[0]$ is a quasi-isomorphism by Lemma 1.1.1.2, the morphism $N_{X/Q_0} \rightarrow \Omega_{Q_0/S_0}^1 \otimes_{\mathcal{O}_{Q_0}} \mathcal{O}_X$ of locally free \mathcal{O}_X -modules is an injection and the assertion follows. \square

Lemma 1.2.7. *Let \mathcal{O}_{K_0} be a discrete valuation ring and let $\mathcal{O}_L \rightarrow \mathcal{O}_{L'}$ be a morphism of ramification index e of discrete valuation rings essentially of finite type over \mathcal{O}_{K_0} . Let*

$$(1.29) \quad \begin{array}{ccc} Q'_0 & \longleftarrow & T' = \text{Spec } \mathcal{O}_{L'} \\ \downarrow & & \downarrow \\ Q_0 & \longleftarrow & T = \text{Spec } \mathcal{O}_L \end{array}$$

be a commutative diagram of essential immersions to smooth schemes over $S_0 = \text{Spec } \mathcal{O}_{K_0}$.

1. *Let $\xi \in Q_0$ and $\xi' \in Q'_0$ be the images of the closed points of T and of T' respectively. Then, the following conditions are equivalent:*

- (1) *The diagram (1.29) is essentially cartesian.*
- (2) *The local ring $\mathcal{O}_{Q'_0 \times_{Q_0} \xi, \xi'}$ is of length e .*

2. *Assume that (1.29) is essentially cartesian. Then, the morphism $Q'_0 \rightarrow Q_0$ is flat on a neighborhood of the image of T' and the canonical morphism*

$$(1.30) \quad N_{T/Q_0} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} \rightarrow N_{T'/Q'_0}$$

is an isomorphism.

3. *Assume that (1.29) is essentially cartesian and that $T' \rightarrow T$ is finite. Then, the diagram (1.29) is cartesian on a neighborhood of T' .*

Proof. 1. Let $X \subset Q_0$, $X' \subset Q'_0$ be as in Definition 1.2.5.3 and let $\xi \in X$, $\xi' \in X'$ be the images of the closed points of T, T' . Let $I \subset \mathcal{O}_{Q_0, \xi}$ and $I' \subset \mathcal{O}_{Q'_0, \xi'}$ be the kernels of the morphisms $\mathcal{O}_{Q_0, \xi} \rightarrow \mathcal{O}_L$ and $\mathcal{O}_{Q'_0, \xi'} \rightarrow \mathcal{O}_{L'}$. Then, we have an inclusion $I\mathcal{O}_{Q'_0, \xi'} \subset I'$ of ideals of $\mathcal{O}_{Q'_0, \xi'}$ and the condition (1) is equivalent to the equality $I\mathcal{O}_{Q'_0, \xi'} = I'$. By the surjection

$$\mathcal{O}_{Q'_0 \times_{Q_0} \xi, \xi'} \rightarrow \mathcal{O}_{X' \times_X \xi, \xi'} = \mathcal{O}_{L'} / \mathfrak{m}_L \mathcal{O}_{L'} = \mathcal{O}_{L'} / \mathfrak{m}_{L'}^e,$$

the condition (2) is equivalent to the isomorphism $\mathcal{O}_{Q'_0 \times_{Q_0} \xi, \xi'} \rightarrow \mathcal{O}_{X' \times_X \xi, \xi'}$.

Let $s \in \mathcal{O}_{Q_0, \xi}$ be a lifting of a uniformizer of \mathcal{O}_L and let $\bar{I} \subset A = \mathcal{O}_{Q_0, \xi} / (s)$ and $\bar{I}' \subset A' = \mathcal{O}_{Q'_0, \xi'} / (s)$ be the images of I and I' . Let E and E' be the residue fields of L and of L' . Then, since $E = A / \bar{I}$, $\mathcal{O}_{Q'_0 \times_{Q_0} \xi, \xi'} = A' / \bar{I}'$ and $\mathcal{O}_{X' \times_X \xi, \xi'} = A' / \bar{I}'$, the condition (2) is further equivalent to the equality $\bar{I}' A' = \bar{I}'$.

Since I and I' are generated by regular sequences of $\mathcal{O}_{Q_0, \xi}$ and of $\mathcal{O}_{Q'_0, \xi'}$ that remain to be regular sequences after joining s , the surjections $I \otimes_{\mathcal{O}_{Q_0, \xi}} E \rightarrow \bar{I} \otimes_A E$ and $I' \otimes_{\mathcal{O}_{Q'_0, \xi'}} E' \rightarrow \bar{I}' \otimes_{A'} E'$ are isomorphisms. By Nakayama's lemma, the equality $\bar{I}' A' = \bar{I}'$ is equivalent to the surjectivity of $\bar{I} \otimes_A E' \rightarrow \bar{I}' \otimes_{A'} E'$ and the equality $I\mathcal{O}_{Q'_0, \xi'} = I'$ is equivalent to the surjectivity of $I \otimes_{\mathcal{O}_{Q_0, \xi}} E' \rightarrow I' \otimes_{\mathcal{O}_{Q'_0, \xi'}} E'$ and the assertion follows.

2. Let $X \subset Q_0$ and $X' \subset Q'_0$ be as in Definition 1.2.5.3. Since $T' \rightarrow T$ is flat, after replacing Q'_0 by a neighborhood of the image of T' , we may assume that $X' \rightarrow X$ is flat. Since (1.28) is cartesian on a neighborhood of the image of T' , the morphism $Q' \rightarrow Q$ is also flat on a neighborhood of the image of T' by [10, Chapitre 0, Proposition (15.1.21) b) \Rightarrow a)]. Hence the morphism (1.30) is an isomorphism.

3. By 2., we may assume that $Q' \rightarrow Q$ is flat and quasi-finite. Since the underlying set of a quasi-finite scheme $T \times_Q Q'$ over T is finite, the normal scheme T' is an open subscheme of $T \times_Q Q'$ and the assertion follows. \square

Lemma 1.2.8. *Let \mathcal{O}_K be a discrete valuation ring essentially of finite type over a discrete valuation ring \mathcal{O}_{K_0} . Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings. We consider a commutative diagram*

$$(1.31) \quad \begin{array}{ccc} Q_0 & \longleftarrow & T = \text{Spec } \mathcal{O}_L \\ \downarrow & & \downarrow \\ S_0 & \longleftarrow & S = \text{Spec } \mathcal{O}_K \end{array}$$

where Q_0 is smooth over S_0 and $T \rightarrow Q_0$ is essentially an immersion.

1. *The morphism $T \rightarrow Q = Q_0 \times_{S_0} S$ defined by the diagram (1.31) is an immersion. If $Q_0 \rightarrow S_0$ is separated, then $T \rightarrow Q$ is a closed immersion.*

2. *Let (1.29) be an essentially cartesian diagram such that $T' \rightarrow T$ is defined by a finite morphism of discrete valuation rings. Then, the diagram*

$$(1.32) \quad \begin{array}{ccc} Q' = Q'_0 \times_{S_0} S & \longleftarrow & T' \\ \downarrow & & \downarrow \\ Q = Q_0 \times_{S_0} S & \longleftarrow & T \end{array}$$

is cartesian on a neighborhood of the image of T' and the canonical morphism

$$(1.33) \quad N_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} \rightarrow N_{T'/Q'}$$

is an isomorphism. After replacing Q_0 and Q'_0 by étale neighborhoods, the vertical arrows are finite flat.

Proof. 1. Since T is finite over S and $T \times_S \text{Spec } F \rightarrow Q$ is an immersion for the residue field F , the morphism $T \rightarrow Q$ is an immersion by Nakayama's lemma. If $Q_0 \rightarrow S_0$ is separated, then $Q \rightarrow S$ is also separated and the immersion $T \rightarrow Q$ is a closed immersion since T is finite over S .

2. By Lemma 1.2.7.3, we may assume that (1.29) is cartesian. Then, (1.32) is cartesian and (1.33) is an isomorphism.

Since $T' \rightarrow T$ is flat, the morphism $Q'_0 \rightarrow Q_0$ is quasi-finite and flat on a neighborhood of the image of $T' \rightarrow Q'_0$. Hence $Q'_0 \rightarrow Q_0$ is finite and flat on étale neighborhoods. \square

Proposition 1.2.9. *Let $S_0 = \text{Spec } \mathcal{O}_{K_0}$ be a discrete valuation ring with perfect residue field k . Let \mathcal{O}_L be a discrete valuation ring essentially of finite type over \mathcal{O}_{K_0} .*

1. *There exists a divisorial essential immersion $T \rightarrow Q_0$ over S_0 ,*

2. Let $T \rightarrow Q_0$ be a divisorial essential immersion. Then, on a neighborhood of the image of T , there exists a divisor $D_0 \subset Q_0$ smooth over S_0 such that D_0 meets T transversally at the image $\xi \in Q_0$ of the closed point of T and that ξ is the generic point of $D_0 \times_{S_0} k$.

3. Let

$$(1.34) \quad \begin{array}{ccccc} S'_0 & \longleftarrow & Q'_0 & \longleftarrow & T' \\ \downarrow & & & & \downarrow \\ S_0 & \longleftarrow & Q_0 & \longleftarrow & T \end{array}$$

be a commutative diagram of schemes such that the rectangle is defined by morphisms of discrete valuation rings. Assume that T and T' are essentially of finite type over S_0 and S'_0 respectively and that the right horizontal arrows are divisorial essential immersions to smooth schemes.

Assume that the ramification index $e_{K'_0/K_0}$ is 1. Then, after replacing Q'_0 by an étale neighborhood, there exists a morphism $Q'_0 \rightarrow Q_0$ of schemes such that the diagram (1.34) with $Q'_0 \rightarrow Q_0$ inserted is commutative and its right square is essentially cartesian.

Proof. 1. We may assume that $\mathcal{O}_L = \mathcal{O}_{X,\xi}$ is a local ring of a normal flat scheme X of finite type over S_0 . Let $t \in \mathcal{O}_L$ be a uniformizer and let $v_1, \dots, v_n \in \mathcal{O}_L$ be liftings of a transcendental basis of the residue field $F = \mathcal{O}_L/\mathfrak{m}_L$ over k such that F is a finite separable extension of $k(\bar{v}_1, \dots, \bar{v}_n)$. After shrinking X if necessary, define a morphism $X \rightarrow \mathbf{A}_{S_0}^{n+1}$ by t, v_1, \dots, v_n . Then, $X \rightarrow \mathbf{A}_{S_0}^{n+1}$ is unramified at ξ . Hence by [10, Chapitre 4, Corollaire (18.4.7)] after further shrinking X if necessary, there exists a scheme Q_0 étale over $\mathbf{A}_{S_0}^{n+1}$ containing X as an effective Cartier divisor.

2. Let $\xi \in X \subset Q_0$ be as in Definition 1.2.5.2. The local ring $\mathcal{O}_{Q_0,\xi}$ is a regular local ring of dimension 2. Let $f \in \mathcal{O}_{Q_0,\xi}$ be a generator of the divisor X and $t_0 \in \mathcal{O}_{K_0}$ be a uniformizer. Then, the classes of f and of t_0 in $\mathfrak{m}_\xi/\mathfrak{m}_\xi^2$ are non-zero. Hence, there exists an element $t \in \mathfrak{m}_\xi$ such that $\mathfrak{m}_\xi = (f, t) = (t_0, t)$.

Let $D_0 \subset Q_0$ be the divisor defined by t . Then, by $\mathfrak{m}_\xi = (t_0, t)$, the divisor $D_0 \times_{S_0} k \subset Q_0 \times_{S_0} k$ is regular and ξ is the generic point of $D_0 \times_{S_0} k$, on a neighborhood of ξ . Since k is perfect, $D_0 \times_{S_0} k$ is smooth over k and $D_0 \subset Q_0$ is smooth over S_0 on a neighborhood of ξ . Further, by $\mathfrak{m}_\xi = (f, t)$, the divisor D_0 meets T transversally at the closed point ξ .

3. Let $\xi \in X \subset Q_0$ and $\xi' \in X' \subset Q'_0$ be as in Definition 1.2.5.2. Let $t_0 \in \mathcal{O}_{K_0} \subset \mathcal{O}_{K'_0}$ be a common uniformizer and let $f \in \mathfrak{m}_\xi \subset \mathcal{O}_{Q_0,\xi}$ and $f' \in \mathfrak{m}_{\xi'} \subset \mathcal{O}_{Q'_0,\xi'}$ be elements defining $X \subset Q_0$ and $X' \subset Q'_0$ on neighborhoods of ξ and ξ' respectively. Let $s \in \mathfrak{m}_\xi$ and $s' \in \mathfrak{m}_{\xi'}$ be elements such that $(f, s) = (t_0, s) = \mathfrak{m}_\xi$ and $(f', s') = (t_0, s') = \mathfrak{m}_{\xi'}$ as in the proof of 2.

Let $e = e_{L'/L}$ be the ramification index. Since $s \in \mathfrak{m}_\xi$ and $s' \in \mathfrak{m}_{\xi'}$ are liftings of uniformizers of \mathcal{O}_L and of $\mathcal{O}_{L'}$, there exists a unit $u \in \mathcal{O}_{Q'_0,\xi'}^\times$ such that $s = us'^e$ in $\mathcal{O}_{L'}$. Then, since $Q_0 \rightarrow S_0$ is smooth and since $T' \rightarrow Q'_0$ is essentially an immersion, after replacing Q'_0 by an étale neighborhood if

necessary, there exists a morphism $Q'_0 \rightarrow Q_0$ that makes the diagram (1.34) commutative and satisfies $s \mapsto us'^e$. Then since $\mathcal{O}_{Q'_0 \times_{Q_0} \xi, \xi'} = \mathcal{O}_{Q'_0, \xi'}/(f, s) = \mathcal{O}_{Q'_0, \xi'}/(f', s'^e) = \mathcal{O}_{L'}/(s'^e)$ is of length e , the right square is essentially cartesian by Lemma 1.2.7.1 (2) \Rightarrow (1). \square

Proposition 1.2.10. *Let \mathcal{O}_{K_0} be a discrete valuation ring with perfect residue field k and let*

$$\begin{array}{ccc} \mathcal{O}_{K'} & \longrightarrow & \mathcal{O}_{L'} \\ \uparrow & & \uparrow \\ \mathcal{O}_K & \longrightarrow & \mathcal{O}_L \end{array}$$

be a commutative diagram of morphisms of discrete valuation rings essentially of finite type over \mathcal{O}_{K_0} such that the horizontal arrows are finite. Let

$$(1.35) \quad \begin{array}{ccc} Q'_0 & \longleftarrow & T' = \text{Spec } \mathcal{O}_{L'} \\ \downarrow & & \downarrow \\ Q_0 & \longleftarrow & T = \text{Spec } \mathcal{O}_L \end{array}$$

be an essentially cartesian diagram of essential immersions to smooth separated schemes over $S_0 = \text{Spec } \mathcal{O}_{K_0}$. Set $S = \text{Spec } \mathcal{O}_K$, $S' = \text{Spec } \mathcal{O}_{K'}$ and let

$$\begin{array}{ccc} Q' = Q'_0 \times_{S_0} S' & \longleftarrow & T' \\ \downarrow & & \downarrow \\ Q = Q_0 \times_{S_0} S & \longleftarrow & T \end{array}$$

be the induced commutative diagram of immersions.

1. The morphism $T' \rightarrow T \times_{Q_0} Q'_0$ defined by (1.35) is formally étale and induces a formally étale morphism $T' \times_S S' \rightarrow T \times_Q Q' = T \times_{Q_0} Q'_0 \times_S S'$.

2. Assume that $T \rightarrow Q_0$ and $T' \rightarrow Q'_0$ are separated and let $\mathcal{J} \subset \mathcal{O}_{S' \times_S S'}$ be the ideal defining the diagonal $S' \rightarrow S' \times_S S'$ and let $\mathcal{I}_T \subset \mathcal{O}_Q$ and $\mathcal{I}_{T'} \subset \mathcal{O}_{Q'}$ be the ideals defining the closed subschemes $T \subset Q$ and $T' \subset Q'$. Let $m \geq 0$ and $q \geq 2$ be integers and set $d' = \dim \Omega_{F'/k}^1$. Let A denote the quotient ring $\mathcal{O}_{K'}/\mathfrak{m}_{K'}^m[X_0, \dots, X_{d'}]/(X_0, \dots, X_{d'})^q$. Then, the following conditions are equivalent:

- (1) We have $\mathcal{I}_T \mathcal{O}_{Q'} \subset \mathfrak{m}_{K'}^m \mathcal{I}_{T'} + \mathcal{I}_{T'}^q$.
- (2) The closed subscheme $Y \subset S' \times_S S'$ defined by the ideal

$$\mathcal{I}_Y = \mathfrak{m}_{K'}^m \mathcal{O}_{S' \times_S S'} + \mathcal{J}^q \subset \mathcal{O}_{S' \times_S S'}$$

is isomorphic to $\text{Spec } A$.

Note that the condition (2) does not depend on T or S_0 .

Proof. 1. Let $X \subset Q_0$ and $X' \subset Q'_0$ be as in Definition 1.2.5.3. We consider the commutative diagram

$$\begin{array}{ccccc} Q'_0 & \longleftarrow & X' & \longleftarrow & T' \\ \downarrow & & \downarrow & & \downarrow \\ Q_0 & \longleftarrow & X & \longleftarrow & T. \end{array}$$

Since (1.35) is assumed essentially cartesian, we may assume that the left square is cartesian. Since the right horizontal arrows are formally étale, the morphism $T' \rightarrow T \times_{Q_0} Q'_0$ is formally étale. Since $T \times_Q Q' = T \times_{Q_0} Q'_0 \times_S S'$, the morphism $T' \times_S S' \rightarrow T \times_Q Q'$ is formally étale.

2. Since $T \rightarrow Q_0$ and $T' \rightarrow Q'_0$ are separated, the immersions $T \rightarrow Q$ and $T' \rightarrow Q'$ are closed immersions by Lemma 1.2.8.1. Let N be the length of the ring A and $e = e_{L'/K'}$ be the ramification index. Let $Z \subset Q'$ be the closed subscheme defined by the ideal $\mathcal{I}_Z = \mathfrak{m}_{K'}^m \mathcal{O}_{Q'} + \mathcal{I}_{T'}^q \subset \mathcal{O}_{Q'}$. Since $Q' \rightarrow S'$ is smooth of relative dimension $d' + 1$ and since $T' \rightarrow Q'$ is a regular immersion, the length of \mathcal{O}_Z is Ne . We show that, for the fiber product $T' \times_{S'} Y$ with respect to the restriction of the first projection, the both conditions (1) and (2) are equivalent to the condition that $\mathcal{O}_{T' \times_{S'} Y}$ is of length Ne .

Since $\mathcal{I}_T \mathcal{O}_{Q'} \subset \mathcal{I}_{T'}$ and $\mathfrak{m}_{K'}^m \mathcal{I}_{T'} + \mathcal{I}_{T'}^q = (\mathfrak{m}_{K'}^m \mathcal{O}_{Q'} + \mathcal{I}_{T'}^q) \cap \mathcal{I}_{T'}$, the condition (1) is equivalent to the inclusion $\mathcal{I}_T \mathcal{O}_{Q'} \subset \mathfrak{m}_{K'}^m \mathcal{O}_{Q'} + \mathcal{I}_{T'}^q = \mathcal{I}_Z$. This means that the closed immersion $T \times_Q Z \rightarrow Z$ is an isomorphism. Since \mathcal{O}_Z is of length Ne , the condition (1) is equivalent to the condition that $\mathcal{O}_{T \times_Q Z}$ is of length Ne . To show that this is equivalent to the condition that $\mathcal{O}_{T' \times_{S'} Y}$ is of length Ne , it suffices to show that the formally étale morphism $T' \times_S S' \rightarrow T \times_Q Q'$ induces an isomorphism $T' \times_{S'} Y \rightarrow T \times_Q Z$ of closed subschemes.

The closed subscheme T' of $T' \times_S S'$ and of $T \times_Q Q'$ is defined by the pull-backs of $\mathcal{J} \subset \mathcal{O}_{S' \times_S S'}$ and of $\mathcal{I}_{T'} \subset \mathcal{O}_{Q'}$ respectively. Since the morphism $T' \times_S S' \rightarrow T \times_Q Q'$ is formally étale by 1, this induces an isomorphism on the closed subschemes defined by the pull-backs of \mathcal{J}^q and of $\mathcal{I}_{T'}^q$ respectively. Hence by $\mathcal{I}_Y = \mathfrak{m}_{K'}^m \mathcal{O}_{S' \times_S S'} + \mathcal{J}^q \subset \mathcal{O}_{S' \times_S S'}$ and $\mathcal{I}_Z = \mathfrak{m}_{K'}^m \mathcal{O}_{Q'} + \mathcal{I}_{T'}^q \subset \mathcal{O}_{Q'}$, the morphism $T' \times_S S' \rightarrow T \times_Q Q'$ induces an isomorphism $T' \times_{S'} Y = (T' \times_S S') \times_{S' \times_S S'} Y \rightarrow T \times_Q Z = (T \times_Q Q') \times_{Q'} Z$. Thus the condition (1) is equivalent to the condition that $\mathcal{O}_{T' \times_{S'} Y}$ is of length Ne as required.

Since $\Omega_{S'/S}^1 = N_{S'/S' \times_S S'}$ is generated by at most $d' + 1$ elements, there exists a closed immersion $Y \rightarrow \text{Spec } A$. Hence the condition (2) is equivalent to the condition that the local ring \mathcal{O}_Y is of length N . Since $T' \rightarrow S'$ is faithfully flat, this is further equivalent to the condition that $\mathcal{O}_{T' \times_{S'} Y}$ is of length Ne . \square

Example 1.2.11. Let \mathcal{O}_K be a henselian discrete valuation ring with residue field F of characteristic $p > 0$. Let $t \in \mathcal{O}_K$ be a uniformizer and $u_1, \dots, u_n \in \mathcal{O}_K$ be a lifting of a p -basis of F . Let $m = qm' \geq 1$ be an integer divisible by a power $q > 1$ of p and $\mathcal{O}_{K'}$ be the henselization of

$$\mathcal{O}_K[v_0, \dots, v_n, w_1, \dots, w_n, t'] / (t - v_0 t'^m, u_i - (w_i^m + v_i t'^m); i = 1, \dots, n)$$

at the prime ideal (t') .

Then, m and q satisfies the condition (2) in Proposition 1.2.10. In fact, set $R = \mathcal{O}_K[T', V_0, \dots, V_n, W_1, \dots, W_n]$ and define a formally unramified morphism $S' = \text{Spec } \mathcal{O}_{K'} \rightarrow \mathbf{A}_S^{2n+2} = \text{Spec } R$ by $t', v_0, \dots, v_n, w_1, \dots, w_n \in \mathcal{O}_{K'}$. Then its base change $S' \times_S S' \rightarrow \mathbf{A}_{S'}^{2n+2}$ defines a formally étale morphism to the closed subscheme defined by the ideal

$$I = (v_0 t'^m \otimes 1 - 1 \otimes V_0 T'^m, (w_i^m + v_i t'^m) \otimes 1 - 1 \otimes (W_i^m + V_i T'^m); i = 1, \dots, n) \subset R.$$

Let

$$J = (v_i - V_i; i = 0, \dots, n, w_i - W_i; i = 1, \dots, n, t' - T') \subset R$$

denote the ideal defining the diagonal section $S' \subset S' \times_S S' \rightarrow \mathbf{A}_{S'}^{2n+2}$. Then, by the congruence $a^m - b^m \equiv (a^{m'} - b^{m'})^q \pmod{p}$ and $p, t'^m \in \mathfrak{m}_K \mathcal{O}_{K'} = \mathfrak{m}_{K'}^m$, we have $I \subset J^q + \mathfrak{m}_{K'}^m R$ and the condition (2) is satisfied. \square

1.3. Differentials and conormal modules.

Lemma 1.3.1. *Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings such that L is a finite separable extension of K and let E be the residue field of L . Set $S = \text{Spec } \mathcal{O}_K$ and $T = \text{Spec } \mathcal{O}_L$.*

1. *The morphisms $\text{Spec } E \rightarrow T \rightarrow S$ define an injection*

$$(1.36) \quad \text{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E) \rightarrow H_1(L_{E/S})$$

of E -vector spaces.

2. *Let $T \rightarrow Q$ be an immersion to a smooth scheme Q over S . Then, the distinguished triangle $L_{Q/S} \otimes_{\mathcal{O}_Q}^L \mathcal{O}_T \rightarrow L_{T/S} \rightarrow L_{T/Q} \rightarrow$ defines an injection*

$$(1.37) \quad \text{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E) \rightarrow N_{T/Q} \otimes_{\mathcal{O}_T} E$$

of E -vector spaces. If $T \rightarrow Q$ is minimal, then (1.37) is an isomorphism.

3. *Let $K \subset M \subset L$ be an intermediate extension and let $T \rightarrow X = \text{Spec } \mathcal{O}_M \rightarrow S$ be the corresponding morphisms. Then, the associated morphism*

$$(1.38) \quad \text{Tor}_1^{\mathcal{O}_M}(\Omega_{\mathcal{O}_M/\mathcal{O}_K}^1, E) \rightarrow \text{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, E)$$

is an injection.

Proof. 1. The morphisms $\text{Spec } E \rightarrow T \rightarrow S$ define a distinguished triangle

$$(1.39) \quad L_{T/S} \otimes_{\mathcal{O}_T}^L E \rightarrow L_{E/S} \rightarrow L_{E/T} \rightarrow$$

of cotangent complexes. Since L is assumed to be a separable extension of K , the canonical morphism $L_{T/S} \rightarrow \Omega_{T/S}^1[0]$ is a quasi-isomorphism by Lemma 1.2.6.4. By this and the quasi-isomorphism $L_{E/T} \rightarrow N_{E/T}[1]$, we obtain an exact sequence

$$(1.40) \quad 0 \rightarrow \text{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E) \rightarrow H_1(L_{E/S}) \rightarrow N_{E/T} \rightarrow \Omega_{T/S}^1 \otimes_{\mathcal{O}_T} E \rightarrow \Omega_{E/F}^1 \rightarrow 0$$

of E -vector spaces and the assertion follows.

2. By Lemma 1.2.6.4, the canonical morphism $L_{T/S} \rightarrow \Omega_{T/S}^1[0]$ is a quasi-isomorphism. Hence the distinguished triangle $L_{Q/S} \otimes_{\mathcal{O}_Q}^L \mathcal{O}_T \rightarrow L_{T/S} \rightarrow L_{T/Q} \rightarrow$ defines an exact sequence

$$(1.41) \quad 0 \rightarrow N_{T/Q} \rightarrow \Omega_{Q/S}^1 \otimes_{\mathcal{O}_Q} \mathcal{O}_T \rightarrow \Omega_{T/S}^1 \rightarrow 0$$

of \mathcal{O}_T -modules. This defines an exact sequence

$$(1.42) \quad 0 \rightarrow \text{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E) \rightarrow N_{T/Q} \otimes_{\mathcal{O}_T} E \rightarrow \Omega_{Q/S}^1 \otimes_{\mathcal{O}_T} E \rightarrow \Omega_{T/S}^1 \otimes_{\mathcal{O}_T} E \rightarrow 0$$

of E -vector spaces.

If $T \rightarrow Q$ is minimal, the right morphism $\Omega_{Q/S}^1 \otimes_{\mathcal{O}_T} E \rightarrow \Omega_{T/S}^1 \otimes_{\mathcal{O}_T} E$ is an isomorphism and hence the left morphism $\mathrm{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E) \rightarrow N_{T/Q} \otimes_{\mathcal{O}_T} E$ is an isomorphism.

3. By the distinguished triangle $L_{X/S} \otimes_{\mathcal{O}_X}^L \mathcal{O}_T \rightarrow L_{T/S} \rightarrow L_{T/X} \rightarrow$ and Lemma 1.2.6.3 applied to $L_{T/X}$, we obtain an injection (1.38). \square

We make Lemma 1.3.1 explicit in the case where \mathcal{O}_L is generated by one element over \mathcal{O}_K . Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings and assume that $L \supsetneq K$ is a separable extension. Assume that the residue field E of L is a purely inseparable extension of degree q of the residue field F of K and let $e = e_{L/K}$ be the ramification index. Assume that \mathcal{O}_L is generated by one element α over \mathcal{O}_K and let $f \in \mathcal{O}_K[X]$ be the minimal polynomial of α . Since $\bar{f} \in F[X]$ is a power of the minimal polynomial of $\bar{\alpha} \in E = F(\bar{\alpha})$, there exists $\bar{u} \in F$ such that $\bar{f} \equiv (X^q - \bar{u})^e$ and that $X^q - \bar{u} \in F[X]$ is irreducible. If $q = 1$ and $\bar{u} = 0$, then f is an Eisenstein polynomial.

Since the immersion $T = \mathrm{Spec} \mathcal{O}_L \rightarrow Q = \mathrm{Spec} \mathcal{O}_K[X]$ defined by α is minimal, the canonical morphism $\mathrm{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E) \rightarrow N_{T/Q} \otimes_{\mathcal{O}_T} E$ (1.37) is an isomorphism by Lemma 1.3.1.2.

Lemma 1.3.2. *Let the notation be as above and let*

$$(1.43) \quad N_{T/Q} \otimes_{\mathcal{O}_T} E \rightarrow H_1(L_{E/S})$$

be the composition of the inverse of the isomorphism (1.37) with the injection $\mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, E) \rightarrow H_1(L_{E/S})$ (1.36).

1. Assume that $f \in \mathcal{O}_K[X]$ is an Eisenstein polynomial and let $\pi = f(0)$ be the constant term. Then the image of the basis $f \in N_{T/Q} \otimes_{\mathcal{O}_T} E$ by (1.43) is $d\pi$.

2. Assume that $q > 1$ and set $f = (X^q - u)^e + t \cdot h$ for a uniformizer $t \in \mathcal{O}_K$, a lifting $u \in \mathcal{O}_K$ of \bar{u} and $h \in \mathcal{O}_K[X]$ such that $\deg h < qe$. Then, the image of f by (1.43) is $-\tilde{d}u + \bar{h} \cdot dt$ if $e = 1$ and $\bar{h} \cdot dt$ if $e > 1$.

Proof. 1. This follows from $f \equiv \pi \pmod{(X, \pi)^2}$.

2. By the commutative diagram

$$(1.44) \quad \begin{array}{ccc} H_1(L_{E/S}) & \longrightarrow & N_{E/Q} \\ \uparrow & & \uparrow \\ \mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, E) & \xrightarrow{\cong} & N_{T/Q} \otimes_{\mathcal{O}_T} E \end{array}$$

of injections, it suffices to show that $X^q - u \in N_{E/Q}$ is the image of $-\tilde{d}u \in H_1(L_{E/S})$. We consider the commutative diagram

$$(1.45) \quad \begin{array}{ccc} \mathrm{Spec} E & \longrightarrow & Q = \mathrm{Spec} \mathcal{O}_K[X] \\ \downarrow & & \downarrow \\ \mathrm{Spec} F(u^{1/p}) & \longrightarrow & \mathbf{A}_S^1 = \mathrm{Spec} \mathcal{O}_K[V] \end{array}$$

where the lower line is defined by $V \mapsto u^{1/p}$ and the right vertical arrow is defined by $V \mapsto X^{q/p}$. Since $X^q - u$ is the image of the basis $V^p - u$ of $N_{F(u^{1/p})/\mathbf{A}_S^1}$, the assertion follows. \square

Example 1.3.3. Let $K = k(u)((t))$ be the field of formal power series of characteristic 2 and let L be the Artin–Schreier extension of K defined by $X^2 - X = u/t^2$. Then the integer ring \mathcal{O}_L is generated by $\alpha = tX$ over \mathcal{O}_K and its minimal polynomial is $f = T^2 - tT - u$. Hence the class of f is $-\tilde{d}u - \alpha \cdot \tilde{d}t$.

Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings essentially of finite type over \mathcal{O}_{K_0} such that the extensions of discrete valuation fields are separable. We consider a commutative diagram

$$(1.46) \quad \begin{array}{ccc} Q_0 & \longleftarrow & T = \text{Spec } \mathcal{O}_L \\ \downarrow & & \downarrow \\ S_0 & \longleftarrow & S = \text{Spec } \mathcal{O}_K \end{array}$$

where Q_0 is smooth over S_0 and $T \rightarrow Q_0$ is essentially an immersion. We consider the distinguished triangle $L_{Q/Q_0} \otimes_{\mathcal{O}_Q}^L \mathcal{O}_T \rightarrow L_{T/Q_0} \rightarrow L_{T/Q}$ of cotangent complexes associated to morphisms $T \rightarrow Q = Q_0 \times_{S_0} S \rightarrow Q_0$. This defines an exact sequence

$$(1.47) \quad 0 \rightarrow N_{T/Q_0} \rightarrow N_{T/Q} \rightarrow \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_T \rightarrow 0$$

of \mathcal{O}_T -modules. In fact, $L_{S/S_0} \rightarrow \Omega_{S/S_0}^1[0]$ is a quasi-isomorphism by the assumption that K is separable over K_0 and by Lemma 1.2.6.4 and $L_{S/Q_0} \rightarrow N_{S/Q_0}[1]$ is a quasi-isomorphism by Lemma 1.2.6.2.

If $T = S$, then the immersion $S \rightarrow Q = Q_0 \times_{S_0} S$ is a section of a smooth morphism $Q = Q_0 \times_{S_0} S \rightarrow S$ and the exact sequence (1.47) is identified with

$$(1.48) \quad 0 \rightarrow N_{S/Q_0} \rightarrow \Omega_{Q_0/S_0}^1 \otimes_{\mathcal{O}_{Q_0}} \mathcal{O}_S \rightarrow \Omega_{S/S_0}^1 \rightarrow 0$$

by the isomorphism $d: N_{S/Q} \rightarrow \Omega_{Q_0/S_0}^1 \otimes_{\mathcal{O}_{Q_0}} \mathcal{O}_S$.

Proposition 1.3.4. *Let $\mathcal{O}_K \rightarrow \mathcal{O}_L$ be a finite morphism of discrete valuation rings essentially of finite type over a discrete valuation ring \mathcal{O}_{K_0} with perfect residue field k of characteristic $p > 0$. We assume that the extensions of discrete valuation fields are separable. Set $S = \text{Spec } \mathcal{O}_K$, $T = \text{Spec } \mathcal{O}_L$ and $S_0 = \text{Spec } \mathcal{O}_{K_0}$. Let Q_0 be a smooth scheme over S_0 and $T \rightarrow Q_0$ be essentially an immersion. Let $T \rightarrow Q = Q_0 \times_{S_0} S$ be the induced immersion. Then we have a commutative diagram*

$$(1.49) \quad \begin{array}{ccc} \text{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, E) & \xrightarrow{(1.36)} & H_1(L_{E/S}) \\ \downarrow (1.37) & \searrow & \downarrow (1.12) \\ N_{T/Q} \otimes_{\mathcal{O}_L} E & \xrightarrow{(1.47)} & \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_K} E \end{array}$$

where the slant arrow is minus of the boundary mapping defined by the exact sequence $0 \rightarrow \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_T \rightarrow \Omega_{T/S_0}^1 \rightarrow \Omega_{T/S}^1 \rightarrow 0$.

Proof. The morphisms $\text{Spec } E \rightarrow T \rightarrow S \rightarrow S_0$ of schemes define a commutative diagram of distinguished triangles

$$(1.50) \quad \begin{array}{ccccccc} L_{S/S_0} \otimes_{\mathcal{O}_S}^L E & \longrightarrow & L_{T/S_0} \otimes_{\mathcal{O}_S}^L E & \longrightarrow & L_{T/S} \otimes_{\mathcal{O}_T}^L E & \longrightarrow & \\ \parallel & & \downarrow & & \downarrow & & \\ L_{S/S_0} \otimes_{\mathcal{O}_S}^L E & \longrightarrow & L_{E/S_0} & \longrightarrow & L_{E/S} & \longrightarrow & \end{array}$$

and the upper right triangle in (1.49) is commutative.

The commutative diagram of schemes

$$\begin{array}{ccccc} T & \longrightarrow & S & \longrightarrow & S_0 \\ & \searrow & \uparrow & & \uparrow \\ & & Q & \longrightarrow & Q_0 \end{array}$$

defines a commutative diagram

$$(1.51) \quad \begin{array}{ccccccc} L_{S/S_0} \otimes_{\mathcal{O}_S} \mathcal{O}_T & \longrightarrow & L_{T/S_0} & \longrightarrow & L_{T/S} & \longrightarrow & \\ \downarrow \simeq & & \downarrow & & \downarrow & & \\ L_{Q/Q_0} \otimes_{\mathcal{O}_Q}^L \mathcal{O}_T & \longrightarrow & L_{T/Q_0} & \longrightarrow & L_{T/Q} & \longrightarrow & \end{array}$$

of distinguished triangles of complexes of \mathcal{O}_T -modules. The canonical morphisms $L_{T/S} \rightarrow \Omega_{T/S}^1[0]$, $L_{S/S_0} \rightarrow \Omega_{S/S_0}^1[0]$ are quasi-isomorphisms by Lemma 1.2.6.4 and $L_{T/Q} \rightarrow N_{T/Q}[1]$ is a quasi-isomorphism by Lemma 1.1.1.3. Hence the diagram (1.51) with $\otimes_{\mathcal{O}_T}^L E$ shows that the lower left triangle in (1.49) is commutative by Lemma 1.1.1.1. \square

The exact sequence (1.47) has the following functoriality. Let

$$(1.52) \quad \begin{array}{ccccccc} T' & \longrightarrow & S' & \longrightarrow & S'_0 & & T' & \longrightarrow & Q'_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & S & \longrightarrow & S_0 & , & T & \longrightarrow & Q_0 \end{array}$$

be commutative diagrams of schemes such that the left one is defined by morphisms of discrete valuation rings and the horizontal arrows in the right one are essentially regular immersions to smooth schemes. We assume that the extensions of discrete valuation fields are separable. Then by the commutative diagram

$$(1.53) \quad \begin{array}{ccccccc} L_{S'/S'_0} \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{T'} & \longrightarrow & L_{T'/Q'_0} & \longrightarrow & L_{T'/Q'} & \longrightarrow & \\ \uparrow & & \uparrow & & \uparrow & & \\ L_{S/S_0} \otimes_{\mathcal{O}_S} \mathcal{O}_{T'} & \longrightarrow & L_{T/Q_0} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & L_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & \end{array}$$

of distinguished triangles and Lemma 1.2.6.4, we have a commutative diagram

$$(1.54) \quad \begin{array}{ccccccc} 0 & \longrightarrow & N_{T'/Q'_0} & \longrightarrow & N_{T'/Q'} & \xrightarrow{d} & \Omega_{S'/S'_0}^1 \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{T'} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & N_{T/Q_0} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & N_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \xrightarrow{d} & \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_{T'} \longrightarrow 0 \end{array}$$

of $\mathcal{O}_{T'}$ -modules. If $S_0 = S'_0, S = S'$ and if the right diagram in (1.52) is essentially cartesian, the vertical arrows are isomorphisms by (1.30) and (1.33). Further if $T = S$, the diagram (1.54) and the exact sequence (1.48) give an isomorphism

$$(1.55) \quad \begin{array}{ccccccc} 0 & \longrightarrow & N_{T'/Q'_0} & \longrightarrow & N_{T'/Q'} & \longrightarrow & \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_{T'} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & N_{S/Q_0} \otimes_{\mathcal{O}_S} \mathcal{O}_{T'} & \longrightarrow & \Omega_{Q_0/S_0}^1 \otimes_{\mathcal{O}_{Q_0}} \mathcal{O}_{T'} & \longrightarrow & \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_{T'} \longrightarrow 0 \end{array}$$

of exact sequences.

Proposition 1.3.5. *Let \mathcal{O}_{K_0} be a discrete valuation ring with perfect residue field and let*

$$\begin{array}{ccc} \mathcal{O}_{K'} & \longrightarrow & \mathcal{O}_{L'} \\ \uparrow & & \uparrow \\ \mathcal{O}_K & \longrightarrow & \mathcal{O}_L \end{array}$$

be a commutative diagram of morphisms of discrete valuation rings essentially of finite type over \mathcal{O}_{K_0} . Assume that the horizontal arrows are finite and that the extensions of discrete valuation fields are separable. Let $S = \text{Spec } \mathcal{O}_K, S' = \text{Spec } \mathcal{O}_{K'},$ and $S_0 = \text{Spec } \mathcal{O}_{K_0}$. Let

$$(1.56) \quad \begin{array}{ccc} Q'_0 & \longleftarrow & T' = \text{Spec } \mathcal{O}_{L'} \\ \downarrow & & \downarrow \\ Q_0 & \longleftarrow & T = \text{Spec } \mathcal{O}_L \end{array}$$

be an essentially cartesian diagram of divisorial essential immersions and let $T \rightarrow Q = Q_0 \times_{S_0} S$ and $T' \rightarrow Q' = Q'_0 \times_{S_0} S'$ be the induced immersions. Let F, F', E and E' be the residue fields of K, K', L and L' and set $d = \dim_F \Omega_{F/k}^1, d' = \dim_{F'} \Omega_{F'/k}^1$.

1. *The commutative diagram (1.54) defines an exact sequence*

$$(1.57) \quad 0 \longrightarrow N_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} \longrightarrow N_{T'/Q'} \longrightarrow \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{T'} \longrightarrow 0.$$

2. *Assume $\dim_{F'} \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} F' = d' + 1$. Then, $\dim_{E'} \text{Tor}_1^{\mathcal{O}_{S'}}(\Omega_{S'/S}^1, E') = d + 1$. The morphism*

$$(1.58) \quad \text{Tor}_1^{\mathcal{O}_{S'}}(\Omega_{S'/S}^1, E') \rightarrow H_1(L_{E'/S})$$

defined by the distinguished triangle $L_{S'/S} \otimes_{\mathcal{O}_{S'}}^L E' \rightarrow L_{E'/S} \rightarrow L_{E'/S'} \rightarrow$ is an isomorphism and the morphism

$$(1.59) \quad H_1(L_{E'/S'}) \rightarrow \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} E'$$

is an injection. The injection

$$(1.60) \quad \mathrm{Tor}_1^{\mathcal{O}_{S'}}(\Omega_{S'/S}^1, E') \rightarrow N_{T/Q} \otimes_{\mathcal{O}_T} E'$$

defined by (1.57) is an isomorphism.

3. We keep assuming $\dim_{F'} \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} F' = d' + 1$. Then, the diagram

$$(1.61) \quad \begin{array}{ccc} \mathrm{Tor}_1^{\mathcal{O}_{S'}}(\Omega_{S'/S}^1, E') & \xrightarrow[(1.58)]{\simeq} & H_1(L_{E'/S}) \\ (1.60) \downarrow \simeq & & \uparrow (1.36) \\ N_{T/Q} \otimes_{\mathcal{O}_T} E' & \xleftarrow{(1.37)} & \mathrm{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E') \end{array}$$

is commutative.

4. For an integer $m \geq 0$, the following conditions are equivalent:

(1) The image of the canonical morphism $N_{T/Q} \rightarrow N_{T'/Q'}$ is a submodule of $\mathfrak{m}_{K'}^m N_{T'/Q'}$ and the morphism $N_{T/Q} \otimes_{\mathcal{O}_L} E' \rightarrow \mathfrak{m}_{K'}^m N_{T'/Q'} \otimes_{\mathcal{O}_L} E'$ induced by $N_{T/Q} \rightarrow \mathfrak{m}_{K'}^m N_{T'/Q'}$ is an injection.

(2) The $\mathcal{O}_{K'}$ -module $\Omega_{S'/S}^1$ is isomorphic to $\mathcal{O}_{K'}^{\oplus(d'-d)} \oplus (\mathcal{O}_{K'}/\mathfrak{m}_{K'}^m)^{\oplus(d+1)}$. If these equivalent conditions are satisfied, we have a canonical isomorphism

$$(1.62) \quad \mathrm{Tor}_1^{\mathcal{O}_{K'}}(\Omega_{S'/S}^1, F') \rightarrow \Omega_{S'/S, \mathrm{tors}}^1 \otimes_{\mathcal{O}_{K'}} \mathfrak{m}_{K'}^m / \mathfrak{m}_{K'}^{m+1}.$$

Note that the condition (2) in 4. is independent of T .

Proof. 1. Since the diagram (1.56) is assumed to be essentially cartesian, the left vertical arrow in (1.54) is an isomorphism by Lemma 1.2.7.2. Hence (1.54) induces an isomorphism

$$\begin{aligned} \mathrm{Coker}(N_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} \rightarrow N_{T'/Q'}) &\rightarrow \mathrm{Coker}(\Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \mathcal{O}_{T'} \rightarrow \Omega_{S'/S'_0}^1 \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{T'}) \\ &= \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{T'} \end{aligned}$$

and we obtain the exact sequence (1.57).

2. Let $S \rightarrow X$ and $S' \rightarrow X'$ be morphisms to irreducible regular schemes of finite type over S_0 inducing isomorphisms on the henselizations and let $X' \rightarrow X$ be a morphism compatible with $S' \rightarrow S$. We may assume that the generic fibers X_{K_0} and X'_{K_0} are smooth over K_0 and that the morphism $X'_{K_0} \rightarrow X_{K_0}$ is smooth. Then, we have $\dim X = d + 1$, $\dim X' = d' + 1$ and hence $\dim_{K'} \Omega_{\mathcal{O}_{K'}/\mathcal{O}_K}^1 \otimes_{\mathcal{O}_{K'}} K' = d' - d$. Thus, the assumption $\dim_{F'} \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} F' = d' + 1$ implies $\dim_{E'} \mathrm{Tor}_1^{\mathcal{O}_{S'}}(\Omega_{S'/S}^1, E') = d + 1$.

Since $L_{E'/S'}$ is acyclic except at degree $[-1, 0]$, the distinguished triangle $L_{S'/S} \otimes_{\mathcal{O}_{S'}}^L E' \rightarrow L_{E'/S} \rightarrow L_{E'/S'} \rightarrow$ defines an exact sequence $0 \rightarrow \mathrm{Tor}_1^{\mathcal{O}_{S'}}(\Omega_{S'/S}^1, E') \rightarrow H_1(L_{E'/S}) \rightarrow H_1(L_{E'/S'}) \rightarrow \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} E'$. By the exact sequence (1.5), we have $\dim_{E'} H_1(L_{E'/S}) \leq d + 1$. Hence the injection (1.58) is an isomorphism and the morphism (1.59) is an injection. Since

$Q_0 \rightarrow S_0$ is of relative dimension $d + 1$, the free \mathcal{O}_T -module $N_{T/Q}$ is of rank $d + 1$. Hence the injection (1.60) induced by (1.57) is an isomorphism.

3. By the functoriality of (1.36), the diagram

$$(1.63) \quad \begin{array}{ccc} \mathrm{Tor}_1^{\mathcal{O}_{S'}}(\Omega_{S'/S}^1, E') & \xrightarrow[\text{(1.58)}]{\simeq} & H_1(L_{E'/S}) \\ \downarrow & \nearrow & \uparrow \text{(1.36)} \\ \mathrm{Tor}_1^{\mathcal{O}_{T'}}(\Omega_{T'/S}^1, E') & \longleftarrow & \mathrm{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E') \end{array}$$

is commutative. Since the slant arrow $\mathrm{Tor}_1^{\mathcal{O}_{T'}}(\Omega_{T'/S}^1, E') \rightarrow H_1(L_{E'/S})$ is an injection by Lemma 1.3.1.1, the left vertical arrow is an isomorphism.

Since the construction of (1.57) is functorial in S' , we obtain a commutative diagram

$$(1.64) \quad \begin{array}{ccccccc} 0 & \longrightarrow & N_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & N_{T'/Q'} & \longrightarrow & \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{T'} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & N_{T'/(Q'_0 \otimes_{S_0} T')} & \longrightarrow & \Omega_{T'/S}^1 \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & N_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & N_{T/(Q_0 \times_{S_0} T)} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} & \longrightarrow & \Omega_{T/S}^1 \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} \longrightarrow 0 \end{array}$$

of exact sequences of $\mathcal{O}_{T'}$ -modules. Since the middle term in the lower line is canonically identified with $\Omega_{Q/S}^1 \otimes_{\mathcal{O}_Q} \mathcal{O}_{T'}$, the diagram

$$(1.65) \quad \begin{array}{ccc} \mathrm{Tor}_1^{\mathcal{O}_{S'}}(\Omega_{S'/S}^1, E') & \xrightarrow{\simeq} & \mathrm{Tor}_1^{\mathcal{O}_{T'}}(\Omega_{T'/S}^1, E') \\ \text{(1.60)} \downarrow \simeq & \nearrow & \uparrow \\ N_{T/Q} \otimes_{\mathcal{O}_T} E' & \xleftarrow{\text{(1.37)}} & \mathrm{Tor}_1^{\mathcal{O}_T}(\Omega_{T/S}^1, E') \end{array}$$

is commutative. By combining (1.63) and (1.65), we obtain (1.61).

4. The exact sequence (1.57) induces an isomorphism

$$(1.66) \quad M = \mathrm{Coker}(N_{T/Q} \otimes_{\mathcal{O}_T} \mathcal{O}_{T'} \rightarrow N_{T'/Q'}) \rightarrow \Omega_{S'/S}^1 \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{T'}.$$

The condition (1) is equivalent to that $M/\mathfrak{m}_{K'}^m M$ is a free $\mathcal{O}_{L'}/\mathfrak{m}_{K'}^m \mathcal{O}_{L'}$ -module of rank $d' + 1$ and that $\mathfrak{m}_{K'}^m M$ is a free $\mathcal{O}_{L'}$ -module of rank $d' - d$. Hence the conditions (1) and (2) are equivalent to each other.

Since $M_{\mathrm{tors}} = N_{T/Q} \otimes_{\mathcal{O}_T} (\mathfrak{m}_{K'}^m / \mathcal{O}_{T'})$, the isomorphism (1.66) induces an isomorphism (1.62). \square

Example 1.3.6. Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of henselian discrete valuation rings in Example 1.2.11. Then $\dim \Omega_{F/k}^1 = n$, $\Omega_{F'/k}^1 = 2n + 1$ and the $\mathcal{O}_{K'}$ -module $\Omega_{\mathcal{O}_{K'}/\mathcal{O}_K}^1$ is isomorphic to $\mathcal{O}_{K'}^{\oplus n+1} \oplus (\mathcal{O}_{K'}/\mathfrak{m}_K \mathcal{O}_{K'})^{\oplus n+1}$. \square

2. SMOOTH GROUP SCHEMES

We first study extensions of vector spaces by finite groups in Section 2.1. The main goal here is a criterion (Proposition 2.1.6) for a finite étale covering of a vector space to be an extension by a finite group. This will be used in the reduction to the perfect residue field case in the first proof of Theorem 4.3.1.

In Section 2.2, we give a criterion (Proposition 2.2.4) for a finite étale covering of a smooth group scheme to be a morphism of group schemes. This will be used to prove the crucial case where the index $r > 1$ is an integer in the second proof of Theorem 4.3.1.

2.1. Additive torsors over vector spaces. Let k be a field of characteristic $p > 0$ and let E be a k -vector space of finite dimension. We consider $E = \text{Spec } S(E^\vee)$ as a smooth algebraic group over k defined by the symmetric algebra over k of the dual space $E^\vee = \text{Hom}_k(E, k)$.

Let $\text{Ext}(E, \mathbf{F}_p)$ denote the group of isomorphism classes of extensions $0 \rightarrow \mathbf{F}_p \rightarrow H \rightarrow E \rightarrow 0$ of smooth commutative group schemes over k . We define a morphism

$$(2.1) \quad E^\vee \rightarrow \text{Ext}(E, \mathbf{F}_p)$$

by sending a linear form $f: E \rightarrow \mathbf{G}_a$ to the class $[H]$ of the extension defined by the commutative diagram

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{F}_p & \longrightarrow & H & \longrightarrow & E \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow f \\ 0 & \longrightarrow & \mathbf{F}_p & \longrightarrow & \mathbf{G}_a & \xrightarrow{F-1} & \mathbf{G}_a \longrightarrow 0 \end{array}$$

of extensions, where $F: \mathbf{G}_a \rightarrow \mathbf{G}_a$ denotes the Frobenius endomorphism.

Since an extension of E by \mathbf{F}_p is an \mathbf{F}_p -torsor over E , by forgetting the group structure, we obtain a canonical morphism

$$(2.3) \quad \text{Ext}(E, \mathbf{F}_p) \rightarrow H^1(E, \mathbf{F}_p).$$

Let $F: S(E^\vee) \rightarrow S(E^\vee)$ denote the absolute Frobenius endomorphism of the symmetric algebra and identify

$$(2.4) \quad H^1(E, \mathbf{F}_p) = \text{Coker}(F - 1: S(E^\vee) \rightarrow S(E^\vee))$$

by the Artin–Schreier sequence $0 \rightarrow \mathbf{F}_p \rightarrow \mathbf{G}_a \xrightarrow{F-1} \mathbf{G}_a \rightarrow 0$. For an integer $m \geq 1$ prime to p , we set

$$(2.5) \quad S^{mp^\infty}(E^\vee) = \varinjlim_{e \in \mathbf{N}} S^{mp^e}(E^\vee)$$

with respect to F .

Proposition 2.1.1. *Let k be a field of characteristic $p > 0$ and let E be a k -vector space of finite dimension.*

1. *For an integer $n \geq 1$, the mapping $F: S^n(E^\vee) \rightarrow S^{pn}(E^\vee)$ is an injection.*

2. The morphism $S(E^\vee) \rightarrow H^1(E, \mathbf{F}_p)$ (2.4) induces an isomorphism

$$(2.6) \quad H^1(k, \mathbf{F}_p) \oplus \bigoplus_{p \nmid m} S^{mp^\infty}(E^\vee) \rightarrow H^1(E, \mathbf{F}_p).$$

3. The morphism (2.3) is an injection and the morphism (2.1) is an isomorphism. The diagram

$$(2.7) \quad \begin{array}{ccc} S(E^\vee) & \xrightarrow{(2.4)} & H^1(E, \mathbf{F}_p) \\ \cup \uparrow & & \uparrow (2.3) \\ E^\vee & \xrightarrow{(2.1)} & \text{Ext}(E, \mathbf{F}_p) \end{array}$$

is commutative.

Proof. 1. Since $S(E^\vee)$ is an integral domain, the endomorphism $F: S(E^\vee) \rightarrow S(E^\vee)$ is an injection.

2. We identify $H^1(E, \mathbf{F}_p) = \text{Coker}(F - 1: S(E^\vee) \rightarrow S(E^\vee))$. Then the right hand side is the direct sum of $H^1(k, \mathbf{F}_p) = \text{Coker}(F - 1: k \rightarrow k)$ and $\text{Coker}(F - 1: \bigoplus_{e=0}^\infty S^{p^e m}(E^\vee) \rightarrow \bigoplus_{e=0}^\infty S^{p^e m}(E^\vee)) = S^{mp^\infty}(E^\vee)$ for $m \geq 1$ prime to p .

3. We give a proof for convenience. By the definitions of (2.4) and (2.1), the diagram is commutative. Since the composition through the upper left is an injection by 1. and 2., it suffices to show the surjectivity of (2.1). Since E is a direct sum of copies of \mathbf{G}_a , we may assume $E = \mathbf{G}_a$.

Since an automorphism of hyperbolic curve has no non-trivial deformation, a connected affine smooth group scheme of dimension 1 over k is isomorphic to \mathbf{G}_a or a form of \mathbf{G}_m . Since there is no non-trivial morphism $\mathbf{G}_m \rightarrow \mathbf{G}_a$ of group schemes, a non-trivial extension of \mathbf{G}_a by \mathbf{F}_p is isomorphic to \mathbf{G}_a . Since a finite étale morphism $\mathbf{G}_a \rightarrow \mathbf{G}_a$ of group schemes of degree p with split kernel is $a(F - b^{p-1})$ for $a, b \in k^\times$, the morphism (2.1) for $E = \mathbf{G}_a$ is a surjection. \square

By the isomorphism (2.1), we identify $\text{Ext}(E, \mathbf{F}_p) = E^\vee$. By the isomorphism (2.6), we identify $H^1(E, \mathbf{F}_p) = H^1(k, \mathbf{F}_p) \oplus \bigoplus_{p \nmid m} S^{mp^\infty}(E^\vee)$. Then, the injection $\text{Ext}(E, \mathbf{F}_p) \rightarrow H^1(E, \mathbf{F}_p)$ (2.3) is induced by $E^\vee \rightarrow S^{p^\infty}(E^\vee)$. The injection $E^\vee \rightarrow S^{p^\infty}(E^\vee)$ is an isomorphism if and only if E is of dimension 1.

Assume that k is separably closed and let $\pi_1(E, 0)_{\text{pro-}p}$ denote the maximum pro- p quotient of the étale fundamental group defined by the base point at the origin 0.

Corollary 2.1.2. *Let $k \rightarrow k'$ be a morphism of separably closed fields of characteristic $p > 0$. Let E and E' be a k -vector space and a k' -vector space of finite dimension respectively and let $E' \rightarrow E_{k'} = E \otimes_k k'$ be a k' -linear morphism.*

1. *The following conditions are equivalent:*

(1) *The morphism of schemes $E' \rightarrow E$ is dominant.*

(2) *The morphism $H^1(E, \mathbf{F}_p) \rightarrow H^1(E', \mathbf{F}_p)$ of étale cohomology is injective.*

(3) The morphism of pro- p fundamental groups $\pi_1(E', 0)_{\text{pro-}p} \rightarrow \pi_1(E, 0)_{\text{pro-}p}$ is surjective.

2. Assume that k is algebraically closed and that $E' \rightarrow E_{k'} = E \otimes_k k'$ is a surjection. Then, the diagram

$$(2.8) \quad \begin{array}{ccc} H^1(E, \mathbf{F}_p) & \longrightarrow & H^1(E', \mathbf{F}_p) \\ \uparrow & & \uparrow \\ \text{Ext}(E, \mathbf{F}_p) & \longrightarrow & \text{Ext}(E', \mathbf{F}_p) \end{array}$$

is cartesian and the horizontal arrows are injections.

Proof. 1. (1) \Leftrightarrow (2): The condition (1) is equivalent to the injectivity of the morphism of symmetric algebras $S(E^\vee) \rightarrow S(E'^\vee)$. Since $k \rightarrow k'$ is an injection, this is equivalent to the injectivity of $S^n(E^\vee) \rightarrow S^n(E'^\vee)$ for every $n \geq 1$. By Proposition 2.1.1.1, this is equivalent to the injectivity of $S^{mp^\infty}(E^\vee) \rightarrow S^{mp^\infty}(E'^\vee)$ for every $m \geq 1$ prime to p . Since k and k' are separably closed, by Proposition 2.1.1.2, this is equivalent (2).

(2) \Leftrightarrow (3): Since $H^1(E, \mathbf{F}_p)$ is identified with the character group $\text{Hom}(\pi_1(E, 0)_{\text{pro-}p}, \mathbf{F}_p)$ it suffices to apply the following Lemma 2.1.3.

2. By taking a splitting, we may assume $E' = E_{k'}$. Taking a basis, we may assume $E = k^n$ and identify $S(E^\vee) = k[X_1, \dots, X_n]$. Then, by Proposition 2.1.1.2, $H^1(E, \mathbf{F}_p)$ is identified with $\bigoplus_m kX^m$ where $m = (m_1, \dots, m_n)$ runs multiindices such that $p \nmid m_i$ for some i and $\text{Ext}(E, \mathbf{F}_p)$ is identified with $\bigoplus_{i=1}^n kX_i$. Hence the assertion follows. \square

Lemma 2.1.3. *Let $G \rightarrow G'$ be a morphism of pro- p groups. The following conditions are equivalent:*

- (1) $G \rightarrow G'$ is a surjection.
- (2) The induced morphism $\text{Hom}(G', \mathbf{F}_p) \rightarrow \text{Hom}(G, \mathbf{F}_p)$ is an injection.

Proof. We may assume that G and G' are finite p -groups. Then, since p -groups are nilpotent, the condition (1) is equivalent to the surjectivity of the morphism $G_{\text{ab}} \rightarrow G'_{\text{ab}}$ of abelianizations by [7, Chapter 1, §3 Proposition 8, Corollary 4]. By Nakayama's lemma, this is equivalent to the surjectivity of $G_{\text{ab}}/G_{\text{ab}}^p \rightarrow G'_{\text{ab}}/G'_{\text{ab}}^p$. This is equivalent to (2). \square

Definition 2.1.4. *Let k be a field of characteristic $p > 0$ and let E be a k -vector space of finite dimension. Let G be a finite group and let $f: H \rightarrow E$ be a G -torsor on E .*

1. We say that the G -torsor H on E is additive if the following conditions are satisfied: There exists an isomorphism $H \rightarrow \mathbf{G}_a^n$ for $n = \dim_k E$ such that $f: H \rightarrow E$ is an étale morphism of group schemes over k and that the action of G on H is the same as the translation by $G = \text{Ker } f$.

2. We define a morphism

$$(2.9) \quad [H]: \text{Hom}(G, \mathbf{F}_p) \rightarrow H^1(E, \mathbf{F}_p)$$

by sending a character $\chi: G \rightarrow \mathbf{F}_p$ to the image of the class $[H] \in H^1(E, G)$ by χ_* : $H^1(E, G) \rightarrow H^1(E, \mathbf{F}_p)$.

If H is additive, the morphism (2.9) induces a morphism

$$(2.10) \quad [H]: \mathrm{Hom}(G, \mathbf{F}_p) \rightarrow E^\vee = \mathrm{Ext}(E, \mathbf{F}_p)$$

sending a character $\chi: G \rightarrow \mathbf{F}_p$ to the linear form $f \in E^\vee$ such that there exists a commutative diagram

$$(2.11) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & E & \longrightarrow & 0 \\ & & \chi \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & \mathbf{F}_p & \longrightarrow & \mathbf{G}_a & \xrightarrow{F-1} & \mathbf{G}_a & \longrightarrow & 0 \end{array}$$

of exact sequences of smooth group schemes over k .

An additive torsor over \mathbf{G}_a is described as follows.

Lemma 2.1.5. *Let k be a field of characteristic $p > 0$ and let $G \subset k$ be a finite subgroup of the additive group.*

1. *The separable polynomial $a = \prod_{\sigma \in G} (X - \sigma) \in k[X]$ is an additive polynomial and the sequence*

$$(2.12) \quad 0 \longrightarrow G \longrightarrow \mathbf{G}_a \xrightarrow{a} \mathbf{G}_a \longrightarrow 0$$

is exact.

2. *Let $\chi: G \rightarrow \mathbf{F}_p$ be a character and let $\tau \in G$ be an element such that $\chi(\tau) = 1$. Let $b = \prod_{\sigma \in \mathrm{Ker} \chi} (X - \sigma) \in k[X]$ be the additive polynomial. Then the morphism $G^\vee = \mathrm{Hom}(G, \mathbf{F}_p) \rightarrow k = \mathrm{Ext}(\mathbf{G}_a, \mathbf{F}_p)$ (2.10) defined by (2.12) maps χ to $1/b(\tau)^p$.*

Proof. 1. Since $G = a^{-1}(0)$, it suffices to show that a is an additive polynomial. We show this by induction on $\dim_{\mathbf{F}_p} G$. If $G = 0$, then $a = X$ is an additive polynomial. Let $\chi: G \rightarrow \mathbf{F}_p$ be a character and let $\tau \in G$ be an element such that $\chi(\tau) = 1$ as in 2. The polynomial $b \in k[X]$ defined in 2 is an additive polynomial by the induction hypothesis. Hence

$$(2.13) \quad a = \prod_{i \in \mathbf{F}_p} b(X - i\tau) = \prod_{i \in \mathbf{F}_p} (b(X) - ib(\tau)) = b(X)^p - b(\tau)^{p-1}b(X)$$

is an additive polynomial.

2. By (2.13), for $b_1 = b/b(\tau) \in k[X]$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & \mathbf{G}_a & \xrightarrow{a} & \mathbf{G}_a & \longrightarrow & 0 \\ & & \chi \downarrow & & \downarrow b_1 & & \downarrow \cdot \frac{1}{b(\tau)^p} & & \\ 0 & \longrightarrow & \mathbf{F}_p & \longrightarrow & \mathbf{G}_a & \xrightarrow{F-1} & \mathbf{G}_a & \longrightarrow & 0 \end{array}$$

and the assertion follows. \square

Proposition 2.1.6. *Let k be a field of characteristic $p > 0$ and let E be a k -vector space of finite dimension. Let G be a finite group and let $f: H \rightarrow E$ be a G -torsor on E . Then, the following conditions (1)–(3) are equivalent to each other:*

(1) *H is additive.*

(2) *There exists a structure of group scheme over k on H such that $f: H \rightarrow E$ is a morphism of group schemes over k and H is connected.*

(3) The group G is an \mathbf{F}_p -vector space and the morphism $[H]: \text{Hom}(G, \mathbf{F}_p) \rightarrow H^1(E, \mathbf{F}_p)$ is an injection to $\text{Ext}(E, \mathbf{F}_p) = E^\vee$.

We use the following to prove the implication (2) \Rightarrow (1).

Lemma 2.1.7. *Let $f: H \rightarrow G$ be a finite étale morphism of smooth connected group schemes over a field k .*

1. H is a central extension of G by the kernel N .
2. If G is abelian, then H is abelian. Further if G is killed by an integer $n \geq 1$, then H is also killed by n .
3. If G is a vector space over k , then H is also isomorphic to a vector space over k .

Proof. 1. By extending k if necessary, we may assume that N is split. For $a \in N$, the composition of the morphism $[\cdot, a]: H \rightarrow H$ defined by the commutator $[\cdot, a]$ with $f: H \rightarrow G$ is the constant morphism and induces a morphism $H \rightarrow N$. Since H is connected, this implies that the morphism $[\cdot, a]: H \rightarrow H$ is also constant. Hence H is a central extension of G by N .

2. Since H is an N -torsor over G , the morphism $H \times N \rightarrow H \times_G H$ defined by $(x, y) \mapsto (x, xy)$ is an isomorphism. Since G is commutative, the pair of morphisms $H \times H \rightarrow H$ defined by $(x, y) \mapsto xy$ and $(x, y) \mapsto yx$ defines a morphism $H \times H \rightarrow H \times_G H$. Since H is connected, the composition with the inverse of the isomorphism $H \times N \rightarrow H \times_G H$ and the projection $H \times N \rightarrow N$ is the constant morphism $H \times H \rightarrow N$ defined by the unit $e \in N$. Hence H is commutative.

Similarly, the morphism $H \rightarrow H$ defined by $x \mapsto x^n$ induces the constant morphism $H \rightarrow N$ defined by the unit $e \in N$.

3. If k is of characteristic 0, the morphism $f: H \rightarrow G$ is an isomorphism. Hence, we may assume that k is of characteristic $p > 0$. Assume that G is a vector space. We show that H is also isomorphic to a vector space by induction on the dimension of N as an \mathbf{F}_p -vector space. We may assume that $N = \mathbf{F}_p$. By the isomorphism (2.1), H is the pull-back of the Artin–Schreier extension $0 \rightarrow \mathbf{F}_p \rightarrow \mathbf{G}_a \rightarrow \mathbf{G}_a \rightarrow 0$ by a projection $G \rightarrow \mathbf{G}_a$ and the assertion follows. \square

Proof of Proposition 2.1.6. (1) \Rightarrow (3): Since G is a subgroup of a vector space H , it is an \mathbf{F}_p -vector space and the image of the morphism $[H]: \text{Hom}(G, \mathbf{F}_p) \rightarrow H^1(E, \mathbf{F}_p)$ is a subgroup of $\text{Ext}(E, \mathbf{F}_p)$. For any non-trivial character $\chi: G \rightarrow \mathbf{F}_p$, the extension $\chi_* H$ is connected and its class is non-trivial.

(3) \Rightarrow (2): Since H is an extension of E by G , it is a group scheme over k . Let $H^\circ \subset H$ be the connected component containing 0. If $\chi: G \rightarrow \mathbf{F}_p$ is a character trivial on the intersection $G^\circ = G \cap H^\circ$, then $\chi_* H$ is a trivial extension of E by \mathbf{F}_p . Hence we obtain $\chi = 0$, $G = G^\circ$ and $H = H^\circ$.

(2) \Rightarrow (1): Since H is connected, by Lemma 2.1.7, H is isomorphic to a vector space and is a central extension of E by the kernel $N = \text{Ker } f$. Further since H is connected, the mapping $G = \text{Aut}(H/E) \rightarrow N$ sending g to $g(0)$ is an isomorphism of groups. Hence, G is an \mathbf{F}_p -vector space and the action of G on H is the same as the translation by $N = G$. \square

Let $k \rightarrow k'$ be a morphism of fields of characteristic $p > 0$ and let E and E' be a k -vector space and a k' -vector space respectively. Let $G' \rightarrow G$ be a morphism of finite groups and H be a G -torsor over E and H' be a G' -torsor over E' . Let $E' \rightarrow E$ be a morphism of schemes induced by a k' -linear morphism $E' \rightarrow E \otimes_k k'$ and

$$\begin{array}{ccc} H' & \longrightarrow & E' \\ \downarrow & & \downarrow \\ H & \longrightarrow & E \end{array}$$

be a commutative diagram of schemes compatible with $G' \rightarrow G$. Then, since the G -torsor $H \times_E E' \rightarrow E'$ is isomorphic to the direct image of the G' -torsor $H' \rightarrow E'$ by $G' \rightarrow G$, the diagram

$$(2.14) \quad \begin{array}{ccc} \mathrm{Hom}(G', \mathbf{F}_p) & \xrightarrow{[H']} & H^1(E', \mathbf{F}_p) \\ \uparrow & & \uparrow \\ \mathrm{Hom}(G, \mathbf{F}_p) & \xrightarrow{[H]} & H^1(E, \mathbf{F}_p) \end{array}$$

is commutative.

Corollary 2.1.8. *Let $k \rightarrow k'$ be a morphism of fields of characteristic $p > 0$ and $G' \rightarrow G$ be an injection of finite p -groups. Let E (resp. E') be a k -vector space (resp. k' -vector space) of finite dimension and H (resp. H') be a connected G -torsor (resp. connected G' -torsor) over E (resp. over E'). Let*

$$(2.15) \quad \begin{array}{ccc} H' & \longrightarrow & E' \\ \downarrow & & \downarrow \\ H & \longrightarrow & E \end{array}$$

be a commutative diagram of schemes such that the left vertical arrow is compatible with $G' \rightarrow G$ and that the right vertical arrow is induced by a k' -linear mapping $E' \rightarrow E \otimes_k k'$.

1. Assume that k is separably closed and that the morphism $E' \rightarrow E$ of schemes is dominant. Then, the injection $G' \rightarrow G$ is an isomorphism.
2. If the G -torsor H is additive, then the G' -torsor H' is also additive.
3. Assume that the k' -linear mapping $E' \rightarrow E \otimes_k k'$ is a surjection and that H is geometrically connected. Then $G' \rightarrow G$ is an isomorphism. The G -torsor H is additive if and only if the G' -torsor H' is additive.

Proof. 1. Let \bar{k}' be a separable closure of k' . By replacing k', H' and G' by $\bar{k}',$ a connected component of the base change $H'_{\bar{k}'}$, and its stabilizer, we may assume that k' is also separably closed. Since H and H' are connected, the commutative diagram (2.15) induces a commutative diagram

$$\begin{array}{ccc} \pi_1(E', 0)_{\mathrm{pro}\text{-}p} & \longrightarrow & G' \\ \downarrow & & \downarrow \\ \pi_1(E, 0)_{\mathrm{pro}\text{-}p} & \longrightarrow & G \end{array}$$

where the horizontal arrows are surjections. The left vertical arrow is also a surjection by the assumption that k and k' are separably closed and that $E' \rightarrow E$ is dominant by Corollary 2.1.2.1. Hence $G' \rightarrow G$ is also a surjection.

2. Since G is an \mathbf{F}_p -vector space and since $G' \rightarrow G$ is an injection, G' is also an \mathbf{F}_p -vector space. By the commutative diagram (2.14), if the image of $[H]$ is a subgroup of E^\vee , the image of $[H']$ is a subgroup of E'^\vee . Since H' is connected, the morphism $[H']$ is an injection. Hence H' is additive by Proposition 2.1.6 (3) \Leftrightarrow (1).

3. We show that $G' \rightarrow G$ is an isomorphism. Since H is geometrically connected, by replacing k and k' by algebraic closures, we may assume that k and k' are algebraically closed, similarly as in the beginning of the proof of 1. If $E' \rightarrow E \otimes_k k'$ is surjective, the composition $E' \rightarrow E \otimes_k k' \rightarrow E$ is also surjective and $G' \rightarrow G$ is an isomorphism by 1.

Since the if part is proved in 2., it suffices to show the only if part. Since G' is an \mathbf{F}_p -vector space and since $G' \rightarrow G$ is an isomorphism, G is also an \mathbf{F}_p -vector space. In the commutative diagram (2.14), the assumption implies that the vertical arrows are injections and hence $[H]$ is an injection if $[H']$ is an injection. By Corollary 2.1.2.2, the image of $[H]$ is a subgroup of E^\vee if the image of $[H']$ is a subgroup of E'^\vee . Hence H is additive by Proposition 2.1.6 (3) \Leftrightarrow (1). \square

2.2. Étale isogenies of smooth group schemes.

Lemma 2.2.1. *Let S be a scheme and X and Y be finite étale schemes over S . Then, the functor $\mathcal{I}som_S(X, Y)$ sending a scheme T over S to the set $\text{Isom}_T(X_T, Y_T)$ of isomorphisms $X_T \rightarrow Y_T$ over T is representable by a finite étale scheme $\text{Isom}_S(X, Y)$ over S .*

Proof. The functor $\mathcal{I}som_S(X, Y)$ is representable by a closed subscheme $\text{Isom}_S(X, Y)$ of the vector bundle associated to the locally free \mathcal{O}_S -module $\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_X)$.

We show that the scheme $\text{Isom}_S(X, Y)$ is finite étale over S . Since the assertion is étale local on S and since the sheaf $\mathcal{I}som_S(X, Y)$ is étale locally constant, the assertion follows. \square

Lemma 2.2.2. *Let $X \rightarrow Y$ be a finite étale morphism of schemes and let S be a normal integral scheme over Y . Let K be the residue field of the generic point of S and let K' be an extension of K . Then, the mapping*

$$(2.16) \quad \text{Mor}_Y(S, X) \longrightarrow \text{Mor}_Y(\text{Spec } K', X)$$

sending a morphism $f: S \rightarrow X$ over Y to the composition $\text{Spec } K' \rightarrow S \rightarrow X$ is an injection. The image of (2.16) is the subset defined by the condition that the image of the induced morphism $\text{Spec}(K' \otimes_K K') \rightarrow X \times_Y X$ is a subset of the diagonal $X \subset X \times_Y X$.

Proof. By replacing $X \rightarrow Y$ by the base change $X \times_Y S \rightarrow S$, we may assume $Y = S$. Since S is normal, the étale scheme X over S is also normal. Further replacing S by $\text{Spec } K$, we may assume $S = \text{Spec } K$. Since $K \rightarrow K'$ is injective, the mapping (2.16) is an injection.

Assume that the image of $\mathrm{Spec}(K' \otimes_K K') \rightarrow X \times_Y X$ is a subset of the diagonal $X \subset X \times_Y X$. By replacing X by the image of $\mathrm{Spec} K' \rightarrow X$ and further K' by $\Gamma(X, \mathcal{O}_X)$, we may assume $\mathrm{Spec} K' = X$. Then, the assumption means that the diagonal $X \rightarrow X \times_Y X$ is an isomorphism. Hence, the morphism $X \rightarrow Y$ is of degree 1 and is an isomorphism. \square

Proposition 2.2.3. *Let G be a smooth connected group scheme over a field k with the group structure $\mu: G \times G \rightarrow G$ and the unit $e \in G(k)$.*

Let $f: H \rightarrow G$ be an étale morphism of schemes over k . Let $\nu: H \times H \rightarrow H$ be a morphism of schemes over k such that the diagram

$$(2.17) \quad \begin{array}{ccc} H \times H & \xrightarrow{\nu} & H \\ f \times f \downarrow & & \downarrow f \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

is commutative and let e' be a k -valued point of $N = H \times_G e$. We consider the following conditions:

(1) *The morphism $\nu: H \times H \rightarrow H$ is a group structure and $e' \in H(k)$ is the unit.*

(2) *The morphism $\tilde{\nu}: H \times H \rightarrow H \times H$ defined by $\tilde{\nu}(x, y) = (\nu(x, y), y)$ is an isomorphism and $\nu(e', e') = e'$.*

Then, we have (1) \Rightarrow (2). If H is geometrically connected over k , we have (2) \Rightarrow (1).

Proof. The implication (1) \Rightarrow (2) is obvious. Assume that H is geometrically connected. The inverse image of the diagonal $G \subset G \times G$ by the isomorphism $\tilde{\mu}: G \times G \rightarrow G \times G$ defined by $\tilde{\mu}(x, y) = (\mu(x, y), y)$ is $e \times G$. The condition (2) implies that the diagram

$$(2.18) \quad \begin{array}{ccc} H \times H & \xrightarrow{\tilde{\nu}} & H \times H \\ f \times f \downarrow & & \downarrow f \times f \\ G \times G & \xrightarrow{\tilde{\mu}} & G \times G \end{array}$$

is cartesian. Hence the isomorphism $\tilde{\nu}$ induces an isomorphism $N \times H \rightarrow H \times_G H$. This induces an isomorphism from $e' \times H$ to the diagonal H since $\nu(e', e') = e'$ and H is connected.

Since (2.17) is commutative, the pair of morphisms $H \times H \times H \rightarrow H$ defined by $\nu(\nu(x, y), z)$ and by $\nu(x, \nu(y, z))$ defines a morphism $H \times H \times H \rightarrow H \times_G H$. Since H is geometrically connected, its composition with the projection $H \times_G H \rightarrow N \times H \rightarrow N$ is constant. Since $\nu(e', e') = e'$, its image is e' and the morphism ν satisfies the associativity law. Similarly, e' satisfies the property of the unit element. Further the composition of the morphisms

$$H \xrightarrow{(e', 1_H)} H \times H \xrightarrow{\tilde{\nu}^{-1}} H \times H \xrightarrow{\mathrm{pr}_1} H$$

gives the inverse $H \rightarrow H$. \square

Proposition 2.2.4. *Let G be a smooth geometrically connected group scheme over a field k with the group structure $\mu: G \times G \rightarrow G$ and the unit $e \in G(k)$.*

Let $f: H \rightarrow G$ be an A -torsor for a finite group A and assume that H is geometrically connected over k . Assume that $N = f^{-1}(e)$ has a k -rational point. Let $\xi: \text{Spec } K \rightarrow G$ and $\eta: \text{Spec } L \rightarrow H$ be the generic points. Let L' be an extension of L and $\eta': \text{Spec } L' \rightarrow H$ be the composition with η .

Let

$$\mu_K: G_K = G \times_{\text{Spec } k} \text{Spec } K \rightarrow G_K$$

be the translation $\cdot \xi$ by the generic point ξ defined as the morphism induced by the composition $\mu \circ (1_G, \xi): G \times_{\text{Spec } k} \text{Spec } K \rightarrow G \times G \rightarrow G$. Similarly for a morphism $\nu: H \times H \rightarrow H$, let

$$\nu_{L'}: H_{L'} = H \times_{\text{Spec } k} \text{Spec } L' \rightarrow H_{L'}$$

be the morphism induced by the composition $\nu \circ (1_H, \eta'): H \times_{\text{Spec } k} \text{Spec } L' \rightarrow H \times H \rightarrow H$.

Let M be the set of group structures $\nu: H \times H \rightarrow H$ of schemes over k such that $f: H \rightarrow G$ is a morphism of group schemes. Let M' be the set of isomorphisms $\nu': H_{L'} \rightarrow H_{L'}$ of schemes over L' such that the diagram

$$\begin{array}{ccc} H_{L'} & \xrightarrow{\nu'} & H_{L'} \\ \downarrow & & \downarrow \\ G_K & \xrightarrow{\mu_K} & G_K \end{array}$$

is commutative. Then, the mapping $M \rightarrow M'$ sending ν to $\nu_{L'}$ is a bijection.

To prove the proposition, it suffices to show the following.

Lemma 2.2.5. *Let the notation be as in Proposition 2.2.4. Let M_1 be the set of morphisms $\nu: H \times H \rightarrow H$ of schemes over k such that the diagram (2.18) is commutative and that $\tilde{\nu}: H \times H \rightarrow H \times H$ defined by $\tilde{\nu}(x, y) = (\nu(x, y), y)$ is an isomorphism.*

1. *The mapping $M_1 \rightarrow M'$ sending ν to $\nu_{L'}$ is canonically identified with the mapping*

$$(2.19) \quad \text{Isom}_{G \times H}(H \times H, (G \times H) \times_G H) \rightarrow \text{Isom}_{G_{L'}}(H_{L'}, G_{L'} \times_G H)$$

where the fiber products are taken with respect to $\mu \circ (1_G \times f): G \times H \rightarrow G$ and its composition with $G_{L'} \rightarrow G \times H$.

2. *The mapping (2.19) a bijection.*

3. *We have $M = M_1$.*

Proof. 1. For $\nu: H \times H \rightarrow H$ such that (2.18) is commutative, in the commutative diagram

$$(2.20) \quad \begin{array}{ccccc} H \times H & \xrightarrow{\tilde{\nu}} & H \times H & \xrightarrow{\text{pr}_1} & H \\ \downarrow & & \downarrow & & \downarrow \\ G \times H & \xrightarrow{(\mu \circ (1_G \times f), \text{pr}_2)} & G \times H & \xrightarrow{\text{pr}_1} & G, \end{array}$$

the right square is cartesian. Hence M_1 is identified with the source of (2.19). Since the diagram

$$(2.21) \quad \begin{array}{ccc} H_{L'} & \xrightarrow{\nu_{L'}} & H_{L'} \\ 1_{H \times \eta'} \downarrow & & \downarrow 1_{H \times \eta'} \\ H \times H & \xrightarrow{\tilde{\nu}} & H \times H \end{array}$$

is cartesian, similarly the set M' is identified with the target of (2.19).

2. Define a finite étale scheme μ^*H over $G \times G$ by the cartesian diagram

$$\begin{array}{ccc} \mu^*H & \longrightarrow & H \\ \downarrow & & \downarrow \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

and consider the scheme $X = \text{Isom}_{G \times G}(H \times G, \mu^*H)$ of isomorphisms. By the commutative diagram (2.20) and the cartesian diagram

$$\begin{array}{ccc} H \times G & \longleftarrow & H \times H \\ \downarrow & & \downarrow \\ G \times G & \longleftarrow & G \times H, \end{array}$$

the mapping (2.19) is identified with the canonical mapping

$$(2.22) \quad \text{Mor}_{G \times G}(G \times H, X) \rightarrow \text{Mor}_{G \times G}(G_{L'}, X).$$

We show the bijectivity of (2.22) by applying Lemma 2.2.2. Let L'' be the function field of $G_{L'}$. Then, by Lemma 2.2.2 applied to the morphisms

$$\begin{array}{ccc} \text{Spec } L'' & & X \\ \downarrow & & \downarrow \\ G_{L'} & \longrightarrow & G \times H \xrightarrow{1_G \times f} G \times G, \end{array}$$

the source and the target of (2.22) are identified with subsets of $\text{Mor}_{G \times G}(\text{Spec } L'', X)$. Hence the mapping (2.22) is an injection. Further, its image is the subset defined by the following condition:

(1) The image of the induced morphism $G_{L'} \times_{G \times H} G_{L'} = G \times \text{Spec}(L' \otimes_L L') \rightarrow X \times_{G \times G} X$ is contained in the diagonal $X \subset X \times_{G \times G} X$.

Since G is geometrically connected and since the diagonal $X \subset X \times_{G \times G} X$ is an open and closed subscheme, the condition (1) is equivalent to the following condition:

(1') The image of $e \times \text{Spec}(L' \otimes_L L') \subset G \times \text{Spec}(L' \otimes_L L') \rightarrow X \times_{G \times G} X$ is contained in the diagonal $X \subset X \times_{G \times G} X$.

It suffices to show that the condition (1') is satisfied for any morphism $\text{Spec } L' \rightarrow X$ over $G \times G$. By Lemma 2.2.2 applied to the diagram

$$\begin{array}{ccc} \text{Spec } L' & & X \\ \downarrow & & \downarrow \\ H & \xrightarrow{(e, f)} & G \times G, \end{array}$$

the condition (1') is equivalent to the condition:

(2) The mapping

$$(2.23) \quad \text{Mor}_{G \times G}(H, X) \rightarrow \text{Mor}_{G \times G}(\text{Spec } L', X)$$

is a bijection.

Since the diagrams

$$\begin{array}{ccccccc} H \times G & \longleftarrow & N \times H & & H \times_G H & & \longrightarrow & H \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ G \times G & \xleftarrow{(e,f)} & H, & & H & \xrightarrow{(e,f)} & G \times G & \xrightarrow{\mu} & G \end{array}$$

are cartesian, the condition (2) means the condition:

(2') The mapping

$$(2.24) \quad \text{Isom}_H(N \times H, H \times_G H) \rightarrow \text{Isom}_{L'}(N_{L'}, H \times_G \text{Spec } L')$$

is a bijection.

Since H is an A -torsor over G , the fiber product $H \times_G H$ is canonically identified with $A \times H$. Since $N(k) \subset H(k) \neq \emptyset$ and H is connected, the two sets in (2.24) are canonically identified with the finite set $\text{Isom}(N, A)$. Hence the mapping (2.19) is a bijection.

3. By Proposition 2.2.3, it suffices to show that, for $\nu: H \times H \rightarrow H$ in M_1 , there exists a unique element $e \in N$ such that $\nu(e, e) = e$. Since such an e is the unit element with respect to the group structure ν , it is unique.

We show the existence. Let $a \in N$ and set $\nu(a, a) = b \in N$. Then, there exists a unique element $\sigma \in A$ such that $b = \sigma(a)$. Then, by Proposition 2.2.3, $\nu' = \sigma^{-1} \circ \nu$ is a group structure and a is the unit. Since σ is a unique automorphism of H over G sending a to b , we have $\sigma(x) = \nu'(b, x)$. Define $e \in N$ by $\nu'(b, e) = a$. Then, since $\nu'(\nu'(b, e), \nu'(b, e)) = \nu'(a, a) = a = \nu'(b, e)$, we obtain $\nu(e, e) = \nu'(b, \nu'(e, e)) = e$.

3. CONSTRUCTION OF FUNCTORS

We recall the geometric construction on which the definition of ramification groups is based in Section 3.3. A functorial construction of finite étale coverings of twisted normal spaces is also recalled. This plays a fundamental role in the proof of Theorem 4.3.1. As a preliminary for the construction in Section 3.3, we recall some basic properties of dilatations in Section 3.1.

We prove in Proposition 3.4.5 in Section 3.4 that a finite covering of a twisted normal space constructed in Section 3.3 satisfies a condition in the criterion (Proposition 2.2.4) to be a morphism of group schemes. A crucial ingredient in the proof of Proposition 3.4.5 is a descent property proved in Proposition 3.2.2 of étaleness of finite morphism in Section 3.2.

3.1. Dilatations.

Definition 3.1.1. *Let X be a scheme, $D \subset X$ be an effective Cartier divisor and $Y \subset X$ be a closed subscheme. Let $j: X - D = U \rightarrow X$ be the open immersion of the complement and let $\mathcal{I}_D, \mathcal{I}_Y \subset \mathcal{O}_X$ denote the ideal sheaves defining $D, Y \subset X$. We define the dilatation $X^{[D:Y]}$ to be the scheme*

affine over X defined by the quasi-coherent sheaf of \mathcal{O}_X -subalgebra of $j_*\mathcal{O}_U$ generated by $\mathcal{I}_D^{-1}\mathcal{I}_Y$.

If $X = \text{Spec } A$ and if Y and D are defined by an ideal $I \subset A$ and a non-zero divisor $a \in A$, we have $X^{[D \cdot Y]} = \text{Spec } A\left[\frac{I}{a}\right]$ for the subring $A\left[\frac{I}{a}\right] \subset A\left[\frac{1}{a}\right]$.

The dilatations have the following functoriality. Let

$$(3.1) \quad \begin{array}{ccccc} D \times_X X' \subset D' \subset X' & \longleftarrow & Y' & & \\ \downarrow & & f \downarrow & & \downarrow \\ D & \longrightarrow & X & \longleftarrow & Y \end{array}$$

be a commutative diagram of schemes such that the horizontal arrows are closed immersions and that $D \subset X$ and $D' \subset X'$ are effective Cartier divisors. Then the pull-back $f^*D = D \times_X X' \subset X'$ is an effective Cartier divisor and the morphism $f: X' \rightarrow X$ is uniquely lifted to a morphism $X'^{[D' \cdot Y']} \rightarrow X^{[D \cdot Y]}$. In particular, if $D \times_X Y$ is an effective Cartier divisor of Y , the closed immersion $Y \rightarrow X$ is uniquely lifted to a closed immersion $Y \rightarrow X^{[D \cdot Y]}$.

Lemma 3.1.2. *Assume that in the diagram (3.1), the divisor D' equals the pull-back f^*D .*

1. *Assume that the diagram (3.1) is cartesian and that f is flat. Then, the canonical morphism $X'^{[D' \cdot Y']} \rightarrow X^{[D \cdot Y]} \times_X X'$ is an isomorphism.*

2. *Assume that the morphism f is smooth and that $Y' \rightarrow Y$ is an isomorphism. Then the canonical morphism $X'^{[D' \cdot Y']} \rightarrow X^{[D \cdot Y]}$ is also smooth and the induced morphism $X'^{[D' \cdot Y']} \times_{X'} D' \rightarrow X^{[D \cdot Y]} \times_X D$ is the projection of a vector bundle.*

Proof. 1. Let $j: X - D = U \rightarrow X$ and $j': X' - D' = U' \rightarrow X'$ denote the open immersions of the complements. Then, since f is flat and $D' = f^*D$, the canonical morphism $j_*\mathcal{O}_U \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \rightarrow j'_*\mathcal{O}_{U'}$ is an isomorphism and induces an isomorphism on the subalgebras.

2. Since the assertion is local on a neighborhood of Y' , we may assume that $X = \text{Spec } A$ and $X' = \text{Spec } A'$ are affine and that there exists $t_1, \dots, t_n \in A'$ such that $X' \rightarrow X'_0 = \mathbf{A}_X^n = \text{Spec } A[T_1, \dots, T_n]$ defined by t_1, \dots, t_n is étale and the composition $Y = Y' \rightarrow X' \rightarrow X'_0 = \mathbf{A}_X^n$ is the 0-section.

Assume that $Y \subset X$ and $D \subset X$ are defined by an ideal $I \subset A$ and a non-zero divisor $a \in A$. For $X'_0 = \mathbf{A}_X^n$ and the 0-section $Y \rightarrow \mathbf{A}_X^n$, we have an isomorphism $X_0^{[D'_0 \cdot Y']} \rightarrow \mathbf{A}_{X^{[D \cdot Y]}}^n = \text{Spec } A\left[\frac{I}{a}\right]\left[\frac{T_1}{a}, \dots, \frac{T_n}{a}\right]$. Since $X'^{[D' \cdot Y']} \rightarrow X_0^{[D'_0 \cdot Y']} \times_{X'_0} X'$ is an isomorphism by 1, the morphism $X'^{[D' \cdot Y']} \rightarrow X^{[D \cdot Y]}$ is smooth.

The inverse images $X'^{[D' \cdot Y']} \times_{X'} D'$ and $X_0^{[D'_0 \cdot Y']} \times_{X'_0} D'_0$ are schemes over Y' and the morphism $X'^{[D' \cdot Y']} \rightarrow X_0^{[D'_0 \cdot Y']}$ induces an isomorphism on the inverse images of Y' . Hence the morphism $X'^{[D' \cdot Y']} \times_{X'} D' \rightarrow X_0^{[D'_0 \cdot Y']} \times_{X'_0} D'_0$ is an isomorphism and the scheme $X'^{[D' \cdot Y']} \times_{X'} D'$ is a vector bundle over $X^{[D \cdot Y]} \times_X D$. \square

Proposition 3.1.3. *Let X be a scheme and let $D \subset X$ be a Cartier divisor. Let $Y \subset X$ be a closed subscheme and let $Z = Y \times_X D$ be the intersection.*

1. *The canonical morphism $X^{[D \cdot Z]} \rightarrow X^{[D \cdot Y]}$ of dilatations is an isomorphism.*
2. *Assume that the immersion $Y \rightarrow X$ is a regular immersion and that $Z \subset Y$ is a Cartier divisor. Let $T_Y X$ and $T_Y X(-D)$ be the normal bundle and its twist by the Cartier divisor D . Then, we have a cartesian diagram*

$$(3.2) \quad \begin{array}{ccc} T_Y X(-D) \times_Y Z & \longrightarrow & X^{[D \cdot Y]} \\ \downarrow & & \downarrow \\ D & \longrightarrow & X. \end{array}$$

Proof. 1. Since the assertion is local on X , we may assume that $X = \text{Spec } A$ and Y and D are defined by $I \subset A$ and $a \in A$. Then, we have $X^{[D \cdot Z]} = \text{Spec } A \left[\frac{I + aA}{a} \right] = \text{Spec } A \left[\frac{I}{a} \right] = X^{[D \cdot Y]}$.

2. The twisted normal bundle $T_Y X(-D)$ is the affine scheme over Y defined by the quasi-coherent \mathcal{O}_Y -algebra $\bigoplus_{q=0}^{\infty} \mathcal{I}_D^{-q} \otimes_{\mathcal{O}_X} \mathcal{I}_Y^q / \mathcal{I}_Y^{q+1}$. We show that the canonical morphism

$$\sum_{q=0}^{\infty} \mathcal{I}_D^{-q} \mathcal{I}_Y^q \otimes_{\mathcal{O}_X} \mathcal{O}_D \rightarrow \bigoplus_{q=0}^{\infty} (\mathcal{I}_D^{-q} \otimes_{\mathcal{O}_X} \mathcal{I}_Y^q / \mathcal{I}_Y^{q+1})$$

induces an isomorphism

$$\sum_{q=0}^{\infty} \mathcal{I}_D^{-q} \mathcal{I}_Y^q \otimes_{\mathcal{O}_X} \mathcal{O}_D \rightarrow \bigoplus_{q=0}^{\infty} (\mathcal{I}_D^{-q} \otimes_{\mathcal{O}_X} \mathcal{I}_Y^q / \mathcal{I}_Y^{q+1}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Z.$$

Since the question is local on X , we may assume that $X = \text{Spec } A$, the closed subscheme Y is defined by the ideal $I \subset A$ generated by a regular sequence $t_1, \dots, t_n \in A$ and the divisor D is defined by $a \in A$ such that t_1, \dots, t_n, a remain to be a regular sequence. Then, the morphism $A[T_1, \dots, T_n] / (aT_i - t_i, i = 1, \dots, n) \rightarrow A \left[\frac{1}{a} \right]$ induces an injection on the graded quotients and hence itself is an injection. Thus we obtain an isomorphism $A[T_1, \dots, T_n] / (aT_i - t_i, i = 1, \dots, n) \rightarrow A \left[\frac{I}{a} \right]$ and this induces an isomorphism $A / (aA + I)[T_1, \dots, T_n] \rightarrow A \left[\frac{I}{a} \right] / aA \left[\frac{I}{a} \right]$. \square

Corollary 3.1.4. *Let X be a scheme and let $D \subset X$ be a Cartier divisor. Let $Y, Y' \subset X$ be closed subschemes such that the immersions are regular immersions. Assume that the intersections $Y \times_X D$ and $Y' \times_X D$ are the same and are Cartier divisors of Y and of Y' respectively. We set $Z = Y \times_X D = Y' \times_X D$ and identify $X^{[D \cdot Y]} = X^{[D \cdot Z]} = X^{[D \cdot Y']}$ and $T_Y X(-D) \times_Y Z = T_Z D(-D) = T_{Y'} X(-D) \times_{Y'} Z$.*

Let $u: T_Y X(-D) \times_Y Z \rightarrow T_{Y'} X(-D) \times_{Y'} Z$ be the composition of the isomorphisms

$$(3.3) \quad \begin{aligned} T_Y X(-D) \times_Y Z &\xrightarrow[(3.2)]{\cong} X^{[D \cdot Y]} \times_X D = X^{[D \cdot Z]} \times_X D \\ &= X^{[D \cdot Y']} \times_X D \xleftarrow[(3.2)]{\cong} T_{Y'} X(-D) \times_{Y'} Z \end{aligned}$$

and $s: Z \rightarrow T_Y X(-D) \times_Y Z$ be the inverse image of the 0-section of $T_{Y'} X(-D) \times_{Y'} Z$ by u .

Then u is the composition of the translation $+s: T_Y X(-D) \times_Y Z \rightarrow T_Y X(-D) \times_Y Z$ with the isomorphisms

$$(3.4) \quad T_Y X(-D) \times_Y Z \xleftarrow{\cong} T_Z D(-D) \xrightarrow{\cong} T_{Y'} X(-D) \times_{Y'} Z.$$

Proof. Since the assertion is local on X , we may assume that $X = \text{Spec } A$ is affine, $Y = \text{Spec } A/I$, $Y' = \text{Spec } A/I'$ and $D = \text{Spec } A/aA$. Then, the assumption $Y \times_X D = Y' \times_X D$ means the equality $I + aA = I' + aA$.

Since $(I + aA)/(I + aA)^2 = (N_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{O}_Z) \oplus (N_{D/X} \otimes_{\mathcal{O}_D} \mathcal{O}_Z)$ and the same equality with I and Y replaced by I' and Y' , the equality $I + aA = I' + aA$ induces an isomorphism

$$(N_{Y/X} \otimes_{\mathcal{O}_Y} \mathcal{O}_Z) \oplus (N_{D/X} \otimes_{\mathcal{O}_D} \mathcal{O}_Z) \rightarrow (N_{Y'/X} \otimes_{\mathcal{O}_{Y'}} \mathcal{O}_Z) \oplus (N_{D/X} \otimes_{\mathcal{O}_D} \mathcal{O}_Z)$$

of locally free \mathcal{O}_Z -modules, preserving the subsheaf $N_{D/X} \otimes_{\mathcal{O}_D} \mathcal{O}_Z$ but not necessarily the direct sum decomposition. Hence the morphism u is the composition of (3.4) with the translation $+s'$ by a section $s': Z \rightarrow T_{Y'} X(-D) \times_{Y'} Z$. By comparing the composition with the 0-section $Z \rightarrow T_Y X(-D) \times_Y Z$, we obtain $s' = s$ and the assertion follows. \square

The naive functoriality above is generalized as follows.

Lemma 3.1.5. *Let*

$$(3.5) \quad \begin{array}{ccc} X' & \longleftarrow & Y' \\ f \downarrow & & \downarrow \\ X & \longleftarrow & Y \end{array}$$

be a commutative diagram of schemes such that the horizontal arrows are closed immersions. Let $D \subset X$ and $D', D'_1 \subset X'$ be effective Cartier divisors and $q \geq 1$ be an integer such that the defining ideals satisfy $\mathcal{I}_Y \mathcal{O}_{X'} \subset \mathcal{I}_{D'_1} \mathcal{I}_{Y'} + \mathcal{I}_{Y'}^q$, and $\mathcal{I}_D \cdot \mathcal{I}_{D'_1} + \mathcal{I}_{D'}^q \subset \mathcal{I}_D \mathcal{O}_{X'}$.

Then, the morphism $X' \rightarrow X$ is uniquely lifted to $X'^{[D' \cdot Y']} \rightarrow X^{[D \cdot Y]}$.

The naive functoriality corresponds to the case $q = 1$.

Proof. Since the assertion is local, we may assume that $X = \text{Spec } A$ and $X' = \text{Spec } B$ are affine, that $Y \subset X$ and $Y' \subset X'$ are defined by ideals $I \subset A$ and $J \subset B$ and that $D \subset X$ and $D', D'_1 \subset X'$ are defined by non-zero divisors $a \in A$ and $b, c \in B$. By the assumption, we have $IB \subset cJ + J^q$ and a divides bc and b^q . This implies that the image of $\frac{I}{a} \subset A \left[\frac{1}{a} \right]$ in $B \left[\frac{1}{b} \right]$ is contained in $\frac{cJ}{a} + \frac{J^q}{a} \subset \frac{J}{b} + \frac{J^q}{b^q} \subset B \left[\frac{J}{b} \right]$ and the assertion follows. \square

3.2. Normalizations. Let X be a reduced scheme, $j: U \rightarrow X$ be an immersion of normal dense open subscheme and $V \rightarrow U$ be a finite étale morphism of normal schemes. Then, the normalization Y of X in V is the scheme affine over X corresponding to the quasi-coherent \mathcal{O}_X -algebra \mathcal{O}_Y defined as the integral closure of \mathcal{O}_X in $j_*\mathcal{O}_V$. The formation of normalizations is functorial as follows.

Lemma 3.2.1. *Let*

$$\begin{array}{ccccc} X' & \xrightarrow{\supset} & U' & \longleftarrow & V' & \xrightarrow{\subset} & Y' \\ \downarrow & & \downarrow & & \downarrow & & \\ X & \xrightarrow{\supset} & U & \longleftarrow & V & \xleftarrow{\subset} & Y \end{array}$$

be a cartesian diagram of schemes. Assume that X and X' are reduced, U and U' are their normal dense open subschemes, the middle horizontal arrows are finite étale, and Y and Y' are the normalizations of X and X' in V and V' respectively. Then, there exists a unique morphism $Y' \rightarrow Y$ that makes a square in the right of the diagram commutative.

Proof. Let $Z \subset Y \times_X X'$ be the reduced closed subscheme such that the underlying set is the closure of $V \times_U U' \subset Y \times_X X'$. Then Y' equals the normalization of Z in the finite étale scheme V' over $V \times_U U'$ and we obtain a morphism $Y' \rightarrow Z \rightarrow Y \times_X X' \rightarrow Y$. \square

Proposition 3.2.2. ([3, Lemma 2.7], [17, Lemma 1.20]) *Let \mathcal{O}_K be a discrete valuation ring and $\eta = \text{Spec } K \in S = \text{Spec } \mathcal{O}_K$ be the generic point. Let X be a smooth scheme over S , let $U \subset X_K$ be a dense open subscheme and let $V \rightarrow U$ be a finite étale morphism. Let Y be the normalization of X in V and let $t: S \rightarrow Y$ be a section such that $t(\eta) \in V$.*

Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a quasi-finite morphism of discrete valuation rings and let $X' \supset U' \leftarrow V'$ be the base change by $S' = \text{Spec } \mathcal{O}_{K'} \rightarrow S$. Let Y' be the normalization of X' in V' and let $t': S' \rightarrow Y'$ be the section induced by t .

Assume that $Y' \rightarrow X'$ is étale on a neighborhood of the image $t'(S') \subset Y'$. Then, $Y \rightarrow X$ is étale on a neighborhood of the image $t(S) \subset Y$.

Proposition 3.2.2 is a special case of [3, Lemma 2.7], [17, Lemma 1.20]. For the sake of convenience, we give a proof.

Proof. By replacing S by its strict localization, we may assume that S and S' are strictly local and that $S' \rightarrow S$ is finite. Let x, y, x', y' be the geometric points of X, Y, X', Y' defined as the images of the closed points by $S \rightarrow Y \rightarrow X$ and $S' \rightarrow Y' \rightarrow X'$ and we consider the commutative diagram

$$(3.6) \quad \begin{array}{ccc} Y_{(y)} & \longleftarrow & Y'_{(y')} \\ \downarrow & & \downarrow \\ X_{(x)} & \longleftarrow & X'_{(x')} = X_{(x)} \times_S S' \end{array}$$

of strict localizations. It suffices to prove that the finite morphism $Y_{(y)} \rightarrow X_{(x)}$ is an isomorphism.

Let A denote the normalization of the fiber product $Y_{(y)} \times_{X_{(x)}} X'_{(x')} = Y_{(y)} \times_S S'$. Since A is finite over the strictly local scheme $X'_{(x')}$ and since the degree of $Y_{(y)}$ over $X_{(x)}$ is the same as the degree of A over $X'_{(x')}$, it suffices to show that $A \rightarrow X'_{(x')}$ is an isomorphism.

Since $t(\eta)$ is a point of V and $V' = V \times_S S'$, the inverse image of $t(\eta) \subset Y_{(y)}$ by the finite morphism $A \rightarrow Y_{(y)}$ of normal schemes equals $t'(\eta') \subset A$ for the generic point $\eta' \in S'$. Hence by the going down theorem, the inverse image of the closed point $y \in Y_{(y)}$ by $A \rightarrow Y_{(y)}$ consists of the unique point y' . Since A is a disjoint union of finitely many strictly local schemes and $Y'_{(y')}$ is one of its component, we have $A = Y'_{(y')}$.

By the assumption that $Y' \rightarrow X'$ is étale on a neighborhood of $y' \in t'(S')$, the morphism $A = Y'_{(y')} \rightarrow X'_{(x')}$ of strictly local schemes is étale and hence is an isomorphism. Thus the assertion follows. \square

3.3. Construction of functors. We briefly recall the construction on which the definition of the filtration by ramification groups is based. The definition itself is recalled in Section 4.1. Let K be a henselian discrete valuation field with residue field F . Let L be a finite separable extension of K with residue field E . Let Q be a smooth scheme over $S = \text{Spec } \mathcal{O}_K$ and let $T = \text{Spec } \mathcal{O}_L \rightarrow Q$ be a closed immersion.

Let $r > 0$ be a rational number. Let K' be a finite extension of K of ramification index e with residue field F' and set $S' = \text{Spec } \mathcal{O}_{K'}$. Assume that $er > 0$ is an integer. Let $Q_{S'}^{[er]} = Q_{S'}^{[erQ_{F'} \cdot T_{S'}]}$ denote the dilatation of the base change $Q_{S'} = Q \times_S S'$ at $T_{S'} = T \times_S S'$ and the pull-back $erQ_{F'}$ of the Cartier divisor $\text{Spec } \mathcal{O}_{K'}/\mathfrak{m}_{K'}^{er} \subset S'$. Since $S' \rightarrow S$ is flat, the immersion $T_{S'} \rightarrow Q_{S'}$ is a regular immersion and the conormal sheaf $N_{T_{S'}/Q_{S'}}$ is the pull-back of $N_{T/S}$. Hence the reduced closed fiber $(Q_{S'}^{[er]} \times_S \text{Spec } F')_{\text{red}}$ denoted abusively by $Q_{F'}^{[er]}$ is canonically identified with the vector bundle over $E' = (E \otimes_F F')_{\text{red}}$ associated to the dual of the free E' -module $\text{Hom}_{E'}(\mathfrak{m}_{K'}^{er}/\mathfrak{m}_{K'}^{er+1} \otimes_{F'} E', N_{T/Q} \otimes_{\mathcal{O}_L} E')$ by Proposition 3.1.3.2. If we assume that E is purely inseparable over F , then E' is also a purely inseparable extension of F' . In practice, we may and will later assume that this condition be satisfied since we are interested in ramification.

Let $Q_{S'}^{(er)} \rightarrow Q_{S'}^{[er]}$ denote the normalization and let $\bar{T}_{S'}$ be the normalization of the fiber product $T_{S'} = T \times_S S'$. By the functoriality of dilatations and normalizations, the immersion $T \rightarrow Q$ defines a commutative diagram

$$(3.7) \quad \begin{array}{ccc} \bar{T}_{S'} & \longrightarrow & T_{S'} \\ \downarrow & & \downarrow \\ Q_{S'}^{(er)} & \longrightarrow & Q_{S'}^{[er]} \end{array}$$

By the reduced fiber theorem [6], there exists a finite separable extension K' such that the geometric closed fiber $Q_{S'}^{(er)} \times_{S'} \text{Spec } \bar{F}$ is reduced. If K' is an extension of a Galois closure of L over K , the normalization $\bar{T}_{S'}$ is a disjoint union of sections, parametrized by morphisms $L \rightarrow K'$ over K . For

such extensions K' , the construction of the normalization $Q_{S'}^{(er)}$ commutes with base change. Let \bar{K} be a separable closure and \bar{F} be the residue field. Then, the geometric closed fiber

$$(3.8) \quad Q_{\bar{F}}^{(r)} = Q_{S'}^{(er)} \times_{S'} \text{Spec } \bar{F}$$

is independent of such $K' \subset \bar{K}$.

Let

$$(3.9) \quad Q_{\bar{F}}^{(r)} \rightarrow Q_{\bar{F}}^{[r]} = (Q_{S'}^{[er]} \times_{S'} \text{Spec } \bar{F})_{\text{red}}$$

denote the canonical morphism of the reduced closed fibers and $T_{\bar{F}} = (T \times_S \text{Spec } \bar{F})_{\text{red}} = \text{Spec } (E \otimes_F \bar{F})_{\text{red}}$, by abuse of notation. Since the formation of $Q_{S'}^{[er]}$ commutes with base change by Lemma 3.1.2.1, $Q_{\bar{F}}^{[r]}$ does not depend on S' either.

Let $\mathfrak{m}_{\bar{K}}^r \supset \mathfrak{m}_{\bar{K}}^{r+}$ denote $\{x \in \bar{K} \mid \text{ord}_K x \geq r\} \supset \{x \in \bar{K} \mid \text{ord}_K x > r\}$ and define a vector bundle $N_{T/Q, \bar{F}}^{(r)}$ over $T_{\bar{F}}$ by

$$(3.10) \quad N_{T/Q, \bar{F}}^{(r)} = \text{Hom}_{E_{\bar{F}, \text{red}}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+} \otimes_{F'} E_{\bar{F}, \text{red}}, N_{T/Q} \otimes_{\mathcal{O}_L} E_{\bar{F}, \text{red}})^\vee.$$

The right hand side is defined as the spectrum of the symmetric algebra of the $E_{\bar{F}, \text{red}}$ -module with \vee removed. If we fix a morphism $i_0: L \rightarrow \bar{K}$, we define a connected component

$$(3.11) \quad N_{T/Q, \bar{F}}^{(r)\circ} = \text{Hom}_{E_{\bar{F}, \text{red}}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+}, N_{T/Q} \otimes_{\mathcal{O}_L} \bar{F})^\vee \subset N_{T/Q, \bar{F}}^{(r)}$$

by the induced morphism $\bar{i}_0: E \rightarrow \bar{F}$.

By Proposition 3.1.3.2, we have a canonical isomorphism

$$(3.12) \quad Q_{\bar{F}}^{[r]} \rightarrow N_{T/Q, \bar{F}}^{(r)}$$

If we fix a morphism $i_0: L \rightarrow \bar{K}$, in the commutative diagram (3.7), the morphism i_0 defines a section $S' \rightarrow \bar{T}_{S'}$. This defines \bar{F} -points in $Q_{\bar{F}}^{(r)} \rightarrow Q_{\bar{F}}^{[r]}$. Let

$$(3.13) \quad Q_{\bar{F}}^{(r)\circ} \rightarrow Q_{\bar{F}}^{[r]\circ}$$

denote the connected components containing the points. The isomorphism (3.12) induces an isomorphism $Q_{\bar{F}}^{[r]\circ} \rightarrow N_{T/Q, \bar{F}}^{(r)\circ}$. In the case where E is purely inseparable over F , we have $Q_{\bar{F}}^{[r]\circ} = Q_{\bar{F}}^{[r]}$, or equivalently $N_{T/Q, \bar{F}}^{(r)\circ} = N_{T/Q, \bar{F}}^{(r)}$.

We give an explicit description in monogenic case.

Lemma 3.3.1. *Let L be a wildly ramified finite Galois extension of K of degree n of Galois group G and extend the normalized valuation ord_K to L . Assume that \mathcal{O}_L is generated by one element α over \mathcal{O}_K . Let $f \in \mathcal{O}_K[X]$ be the minimal polynomial of α and set $f = \prod_{i=1}^n (X - \alpha_i)$ so that $\alpha_n = \alpha$ and $\text{ord}_K(\alpha_i - \alpha_n)$ is increasing. Define an ideal $I = (\alpha_{n-1} - \alpha_n) \subsetneq \mathcal{O}_L$ and let $H = \text{Ker}(G \rightarrow \text{Aut}(\mathcal{O}_L/I)) \subset G$ be the smallest non-trivial lower ramification group. Let E be the residue field of L and $e = e_{L/K}$ be the ramification index.*

1. The mapping $H \rightarrow I \otimes_{\mathcal{O}_L} E$ sending $\sigma \in H$ to $\sigma(\alpha) - \alpha$ is an injection of abelian groups.

2. Define a mapping $\beta: H \rightarrow E$ by $\beta(\sigma) = (\sigma(\alpha) - \alpha)/(\alpha_{n-1} - \alpha_n)$. Then, the polynomial $b = \prod_{\sigma \in H} (Y - \beta(\sigma)) \in E[Y]$ is an additive polynomial and we have an exact sequence

$$0 \longrightarrow H \xrightarrow{\beta} \mathbf{A}_E^1 \xrightarrow{b} \mathbf{A}_E^1 \longrightarrow 0.$$

3. Set $\delta = \prod_{i=1}^{n-1} (\alpha_n - \alpha_i) = f'(\alpha)$, $r = \text{ord}_K \delta + \text{ord}_K(\alpha_{n-1} - \alpha_n) \in \frac{1}{n}\mathbf{Z}$, $m = \text{Card}(G - H)$ and define

$$(3.14) \quad c = \prod_{i=1}^m (\alpha_n - \alpha_i) \cdot (\alpha_{n-1} - \alpha_n)^{\text{Card } H}.$$

Then, we have $\text{ord}_K c = r$.

4. Let $T = \text{Spec } \mathcal{O}_L \rightarrow Q = \mathbf{A}_S^1 = \text{Spec } \mathcal{O}_K[X]$ be the immersion defined by α . Define an isomorphism $\mathbf{A}_E^1 \rightarrow Q^{[r]} = N_{T/Q,E}^{(r)}$ by the basis

$$\frac{f}{c} \in N_{T/Q,E}^{(r)} = \text{Hom}(\mathfrak{m}_L^{er}/\mathfrak{m}_L^{er+1}, N_{T/Q} \otimes_{\mathcal{O}_L} E)$$

sending c to f . Then, there exist a connected component $Q_E^{(r)\circ} \subset Q_E^{(r)}$, an isomorphism $\mathbf{A}_E^1 \rightarrow Q_E^{(r)\circ}$ and a commutative diagram

$$(3.15) \quad \begin{array}{ccc} \mathbf{A}_E^1 & \longrightarrow & Q_E^{(r)\circ} \\ b \downarrow & & \downarrow \\ \mathbf{A}_E^1 & \xrightarrow{f/c} & Q_E^{[r]}. \end{array}$$

Proof. 1. Since H acts trivially on $I \otimes_{\mathcal{O}_L} E$, for $\sigma, \tau \in H$, we have $\sigma\tau(\alpha) - \alpha - ((\sigma(\alpha) - \alpha) + (\tau(\alpha) - \alpha)) = \sigma(\tau(\alpha) - \alpha) - (\tau(\alpha) - \alpha) = 0$ in $I \otimes_{\mathcal{O}_L} E$.

2. Since $\alpha_{n-1} - \alpha_n$ is a basis of $I \otimes_{\mathcal{O}_L} E$, the injection $\beta: H \rightarrow E$ is a morphism of abelian groups by 1. Hence the assertion follows from Lemma 2.1.5.

3. For $i = m + 1, \dots, n - 1$, we have $\text{ord}_K(\alpha_i - \alpha_n) = \text{ord}_K(\alpha_{n-1} - \alpha_n)$.

4. Define a morphism $\mathbf{A}_T^1 = \text{Spec } \mathcal{O}_L[Y] \rightarrow Q = \text{Spec } \mathcal{O}_L[X]$ by sending X to $(\alpha_{n-1} - \alpha_n)Y + \alpha_n$. Then since $f((\alpha_{n-1} - \alpha_n)Y + \alpha_n) = \prod_{i=1}^n ((\alpha_{n-1} - \alpha_n)Y - (\alpha_i - \alpha_n))$ is divisible by c , by the universalities of dilatation and normalization, the morphism $\mathbf{A}_T^1 \rightarrow Q$ is uniquely lifted to an morphism $\mathbf{A}_T^1 \rightarrow Q^{(r)}$.

Since the morphism $\mathbf{A}_T^1 \rightarrow Q^{(r)}$ of normal schemes is an isomorphism on the generic fiber and the composition $\mathbf{A}_T^1 \rightarrow Q^{[r]}$ is quasi-finite, the morphism $\mathbf{A}_T^1 \rightarrow Q^{(r)}$ is an open immersion. Since $f/c \equiv b$, the diagram (3.15) with \circ omitted is commutative. Since the morphism $b: \mathbf{A}_E^1 \rightarrow \mathbf{A}_E^1$ is finite, the open immersion $\mathbf{A}_E^1 \rightarrow Q_E^{(r)}$ is an isomorphism to a connected component. \square

Since the normalization $\bar{T}_{S'}$ is a disjoint union of copies of S' indexed by morphisms $L \rightarrow K'$ over K , if K' contains the conjugates of L , the closed

fiber $\bar{T}_{\bar{F}}$ is canonically identified with the finite set $\text{Mor}_K(L, \bar{K})$. Similarly, $T_{\bar{F}}$ is canonically identified with the finite set $\text{Mor}_F(E, \bar{F})$. The reduced geometric closed fibers of (3.7) define the commutative diagram

$$(3.16) \quad \begin{array}{ccc} \text{Mor}_K(L, \bar{K}) = \bar{T}_{\bar{F}} & \longrightarrow & T_{\bar{F}} = \text{Mor}_F(E, \bar{F}) \\ \downarrow & & \downarrow \\ Q_{\bar{F}}^{(r)} & \longrightarrow & Q_{\bar{F}}^{[r]} \end{array}$$

The construction of (3.16) satisfies the following functoriality.

Proposition 3.3.2. *Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of henselian discrete valuation rings of ramification index e . Let L and L' be finite separable extensions of K and K' and let $L \rightarrow L'$ be a morphism compatible with $K \rightarrow K'$. Fix a morphism $\bar{K} \rightarrow \bar{K}'$ of separable closures compatible with $K \rightarrow K'$. Let*

$$(3.17) \quad \begin{array}{ccc} Q' & \longleftarrow & T' = \text{Spec } \mathcal{O}_{L'} \\ \downarrow & & \downarrow \\ Q & \longleftarrow & T = \text{Spec } \mathcal{O}_L \end{array}$$

be a commutative diagram of schemes such that Q and Q' are smooth schemes over $S = \text{Spec } \mathcal{O}_K$ and $S' = \text{Spec } \mathcal{O}_{K'}$, that the horizontal arrows are closed immersions and that the vertical arrows are compatible with $S' \rightarrow S$. Let $r > 1$ and $r' > 1$ be rational numbers.

1. Let $r' = er$. Then, the diagram (3.17) defines a commutative diagram

$$(3.18) \quad \begin{array}{ccccccc} \bar{T}'_{\bar{F}'} & \longrightarrow & Q'^{(r')}_{\bar{F}'} & \longrightarrow & Q'^{[r']}_{\bar{F}'} & \longrightarrow & T'_{\bar{F}'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{T}_{\bar{F}} & \longrightarrow & Q^{(r)}_{\bar{F}} & \longrightarrow & Q^{[r]}_{\bar{F}} & \longrightarrow & T_{\bar{F}} \end{array}$$

of schemes compatible with the induced morphism $\bar{F} \rightarrow \bar{F}'$ of residue fields of \bar{K} and \bar{K}' .

2. Assume that $S' = S$, $T' = T$, $r = r'$ and that $Q' \rightarrow Q$ is smooth. Then the middle square in (3.18) is cartesian and the middle right vertical arrow is a surjection of vector bundles over $T_{\bar{F}}$.

3. Assume that the discrete valuation rings \mathcal{O}_K and $\mathcal{O}_{K'}$ are essentially of finite type over a discrete valuation ring \mathcal{O}_{K_0} with perfect residue field k and that there exist integers $m \geq 0$ and $q \geq 2$ satisfying the condition (2) in Proposition 1.2.10 such that $r \geq \frac{m}{e} \frac{q}{q-1}$ and $r' = er - m \geq \frac{m}{q-1}$. Assume further that (3.17) is induced by an essentially cartesian diagram

$$(3.19) \quad \begin{array}{ccc} Q'_0 & \longleftarrow & T' \\ \downarrow & & \downarrow \\ Q_0 & \longleftarrow & T \end{array}$$

of essential immersions. Then, the diagram (3.17) defines a commutative diagram (3.18) compatible with the induced morphism $\bar{F} \rightarrow \bar{F}'$ of residue fields of \bar{K} and \bar{K}' as in 1.

Proof. 1. Let

$$\begin{array}{ccc} S'_1 & \longrightarrow & S' \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S \end{array}$$

be a commutative diagram of spectra of henselian discrete valuation rings such that the horizontal arrows are finite of ramification indices e_1 and e'_1 . Assume that $e_1 r$ and $e'_1 r'$ are integers. Then, by the naive functoriality of dilatations, the morphism $Q' \rightarrow Q$ is lifted to a morphism $Q'_{S'_1} \rightarrow Q_{S_1}$. This induces a morphism $Q'_{S'_1} \rightarrow Q_{S_1}$ of normalizations and the assertion follows.

2. By Lemma 1.2.1.3, if $Q' \rightarrow Q$ is smooth, we may assume that there is an étale morphism $Q' \rightarrow Q'_0 = \mathbf{A}_Q^n$ such that the composition $T \rightarrow Q' \rightarrow \mathbf{A}_Q^n$ is the 0-section. Then, in the notation of the proof of 1., we have an isomorphism $Q'_{S'_1} \rightarrow \mathbf{A}_{Q_{S_1}}^n \times_{Q'_0} Q'$ inducing isomorphisms $Q'_{\bar{F}} \rightarrow Q_{0,\bar{F}}$ and $Q'^{[r]}_{\bar{F}} \rightarrow Q_{0,\bar{F}}^{[r]}$ by Lemma 3.1.2.2. Hence the assertion follows.

3. Let $\mathcal{I}_T \subset \mathcal{O}_Q$ and $\mathcal{I}_{T'} \subset \mathcal{O}_{Q'}$ be the ideals defining the closed subschemes $T \subset Q$ and $T' \subset Q'$. Then, by Proposition 1.2.10, we have $\mathcal{I}_T \mathcal{O}_{Q'} \subset \mathfrak{m}_{K'}^m \mathcal{I}_{T'} + \mathcal{I}_{T'}^q$. Hence by the generalized functoriality Lemma 3.1.5 of dilatations, the assertion follows as in 1. \square

Since the left vertical arrow in (3.16) is induced by the immersion $T \rightarrow Q$, its composition with (3.12) is the 0-section $T_{\bar{F}} \rightarrow N_{T/Q,\bar{F}}^{(r)}$ of the vector bundle. The surjectivity of the mappings $\varphi_{\bar{x}}^{D+}: \bar{Y}_{\bar{x}} \rightarrow F_{\bar{x}}^{D+}(Y/X) \rightarrow F_{\bar{x}}^D(Y/X)$ in [18, Proposition 3.1.2 (1)] means exactly that the mappings

$$(3.20) \quad \bar{T}_{\bar{F}} \rightarrow Q_{\bar{F}}^{(r)} \times_{Q_{\bar{F}}^{[r]}} T_{\bar{F}} \rightarrow \pi_0(Q_{\bar{F}}^{(r)})$$

defined by the diagram (3.16) are surjections. This fact is a consequence of the going down theorem.

Corollary 3.3.3. *There exist surjections*

$$(3.21) \quad \bar{T}_{\bar{F}} \rightarrow F^{r+}(L) \rightarrow F^r(L)$$

of finite sets satisfying the following property: Let $T \rightarrow Q$ be an immersion to a smooth scheme over S . Then, there exists a commutative diagram

$$(3.22) \quad \begin{array}{ccccc} \bar{T}_{\bar{F}} & \longrightarrow & Q_{\bar{F}}^{(r)} \times_{Q_{\bar{F}}^{[r]}} T_{\bar{F}} & \longrightarrow & \pi_0(Q_{\bar{F}}^{(r)}) \\ & \searrow & \downarrow & & \downarrow \\ & & F^{r+}(L) & \longrightarrow & F^r(L) \end{array}$$

such that the vertical arrows are bijections.

Proof. Let $T \rightarrow Q$ and $T \rightarrow Q'$ be immersions to smooth schemes over S . Then by Proposition 3.3.2.2, the smooth projections $Q \times_S Q' \rightarrow Q$ and $Q \times_S Q' \rightarrow Q'$ induces bijections of the finite sets compatible with the surjections from $\bar{T}_{\bar{F}}$. Hence the assertion follows. \square

By Proposition 3.3.2.1, the constructions of $F^r(L)$ and $F^{r+}(L)$ are functorial in L .

Definition 3.3.4. *Let*

$$(3.23) \quad F: (\text{Finite separable extensions of } K) \rightarrow (\text{Finite } G_K\text{-sets})$$

denote the fiber functor defined by $F(L) = \bar{T}_{\bar{F}}$. Let $r > 0$ be a rational number.

1. *We define functors*

$$(3.24) \quad F^r, F^{r+}: (\text{Finite separable extensions of } K) \rightarrow (\text{Finite } G_K\text{-sets})$$

sending L to $F^r(L)$ and $F^{r+}(L)$ and surjective morphisms

$$(3.25) \quad F \rightarrow F^{r+} \rightarrow F^r$$

of functors as in Corollary 3.3.3.

2. *Let L be a finite separable extension of K . We say that the ramification is bounded by r (resp. by $r+$) if the mapping $F(L) \rightarrow F^r(L)$ (resp. $F(L) \rightarrow F^{r+}(L)$) is a bijection.*

We identify $F^1(L) = T_{\bar{F}}$ by [1, Proposition 3.7], [18, Proposition 3.3.5]. If L is a finite Galois extension, the Galois group $G = \text{Gal}(L/K)$ acts on the finite G_K -set $F(L) = \text{Mor}_K(L, \bar{K})$ and its quotients $F^r(L)$ and $F^{r+}(L)$ by the functoriality. This action commutes with the G_K -action. If M is a subextension of L and if we fix $L \rightarrow \bar{K}$, the G_K -action factors through the quotient G .

The functor F has the following functoriality. Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of henselian discrete valuation field of ramification index e and fix a morphism $\bar{K} \rightarrow \bar{K}'$ of separable closures. Let F' denote the functor F for finite separable extensions of K' and let $L \rightarrow L'$ be a morphism of finite separable extensions of K and of K' . Then, for rational numbers $r > 0$ and $r' = er$ or $r' = er - m > 0$ as in Proposition 3.3.2, the commutative diagram (3.18) defines a commutative diagram

$$(3.26) \quad \begin{array}{ccccc} F'(L') & \longrightarrow & F'^{r'+}(L') & \longrightarrow & F'^{r'}(L') \\ \downarrow & & \downarrow & & \downarrow \\ F(L) & \longrightarrow & F^{r+}(L) & \longrightarrow & F^r(L). \end{array}$$

Lemma 3.3.5. *Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of henselian discrete valuation rings of ramification index $e \geq 1$. Let $r > 0$ and $r' > 0$ be rational numbers as in Proposition 3.3.2 satisfying one of the following conditions (1) and (2):*

(1) $r' = er$.

(2) *The discrete valuation rings \mathcal{O}_K and $\mathcal{O}_{K'}$ are essentially of finite type over a discrete valuation ring \mathcal{O}_{K_0} with perfect residue field k . There exist*

integers $m \geq 0$ and $q \geq 2$ satisfies the condition (1) in Proposition 1.2.10 such that $r \geq \frac{m}{e} \frac{q}{q-1}$ and $r' = er - m \geq \frac{m}{q-1}$.

Let $L \rightarrow L'$ be a morphism of finite separable extensions of K and of K' . If the ramification of L is bounded by r (resp. $r+$), then ramification of L' is bounded by r' (resp. $r'+$).

Proof. In the commutative diagram (3.26), if the ramification of L over K is bounded by r (resp. by $r+$), then the composition of lower horizontal arrows (resp. the lower left horizontal arrow) is a bijection. Since the left vertical arrow is an injection, the composition of upper horizontal arrows (resp. the upper left horizontal arrow) is an injection and the ramification of L' over K' is bounded by r' (resp. by $r'+$). \square

Proposition 3.3.6. ([1], [18]) *Let K be a henselian discrete valuation field and L be a finite separable extension. Let*

$$(3.27) \quad \begin{array}{ccc} Q & \longleftarrow & T = \text{Spec } \mathcal{O}_L \\ \downarrow & & \downarrow \\ P & \longleftarrow & S = \text{Spec } \mathcal{O}_K \end{array}$$

be a cartesian diagram of schemes over S where the horizontal arrows are closed immersions to smooth schemes over S and the vertical arrows are quasi-finite and flat. Let $r > 0$ be a rational number and let K' be a finite separable extension of ramification index e of K such that er is an integer and that the morphism $Q_{S'}^{(er)} \rightarrow S' = \text{Spec } \mathcal{O}_{K'}$ has reduced geometric fibers.

1. *The following conditions are equivalent:*

- (1) *The ramification of L over K is bounded by $r+$.*
- (2) *The ramification of L over K is bounded by s for every rational number $s > r$.*

If $r > 1$, these conditions are further equivalent to the following condition.

- (3) *The finite morphism $Q_{\bar{F}}^{(r)} \rightarrow Q_{\bar{F}}^{[r]}$ is étale.*

(4) *The quasi-finite morphism $Q_{S'}^{(er)} \rightarrow P_{S'}^{(er)}$ is étale on neighborhood of the closed fibers.*

2. *The following conditions are equivalent:*

- (1) *The ramification of L over K is bounded by r .*
- (2) *The finite morphism $Q_{\bar{F}}^{(r)} \rightarrow Q_{\bar{F}}^{[r]}$ is a split étale covering.*
- (3) *The finite morphism $Q_{\bar{F}}^{(r)} \rightarrow P_{\bar{F}}^{(r)}$ is a split étale covering.*

Proof. 1. The implication (1) \Rightarrow (2) is [18, Lemma 3.1.6]. The implication (2) \Rightarrow (1) is [18, Theorem 3.2.6].

If (3) is satisfied, the diagram (3.16) is cartesian and (1) is satisfied. We show (4) \Rightarrow (3). For an immersion $T \rightarrow Q$ to a smooth scheme over S , there exists a cartesian diagram (3.27) by Lemma 1.2.1.1. By the reduced fiber theorem [6], there exists a finite separable extension K' of ramification index e such that er is an integer and that the morphism $Q_{S'}^{(er)} \rightarrow S' = \text{Spec } \mathcal{O}_{K'}$ has reduced geometric fibers. Since the diagram (3.27) is cartesian, the finite morphism $Q_{\bar{F}}^{[r]} \rightarrow P_{\bar{F}}^{[r]} = P_{\bar{F}}^{(r)}$ is étale. Hence by taking the geometric closed

fiber of $Q_{S'}^{(er)} \rightarrow P_{S'}^{(er)}$, we obtain (3) from (4). Conversely, (3) implies (4) by [10, Théorème (18.10.1)]. The implication (2) \Rightarrow (3) in the case $r > 1$ is proved in [1, Theorem 7.2].

2. We show that (2) and (3) are equivalent to each other. As in the proof of 1 (4) \Rightarrow (3), there exists a cartesian diagram (3.27). Since $Q \rightarrow P$ is flat, the finite morphism $Q_{\bar{F}}^{[r]} \rightarrow P_{\bar{F}}^{[r]}$ is a split étale covering. Hence the conditions (2) and (3) are equivalent to each other.

We show that (1) and (3) are equivalent to each other. As in the proof of 1 (4) \Rightarrow (3), we have a cartesian diagram (3.27) and a finite separable extension K' loc. cit. Since the assertion is étale local, we may assume that the morphism $Q \rightarrow P$ is finite. Hence $Q_{S'}^{(er)} \rightarrow P_{S'}^{(er)}$ is a finite morphism of degree $[L : K]$ and the conditions (1) and (3) are equivalent to each other. \square

For a rational number $r > 0$, let

$$(3.28) \quad \Theta_{L/K, \bar{F}}^{(r)} = \mathrm{Hom}_{E_{\bar{F}, \mathrm{red}}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+} \otimes_{\bar{F}} E_{\bar{F}, \mathrm{red}}, \mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, E_{\bar{F}, \mathrm{red}}))^\vee$$

denote the vector bundle over $T_{\bar{F}}$. If we fix a morphism $i_0: L \rightarrow \bar{K}$, we define a connected component

$$(3.29) \quad \Theta_{L/K, \bar{F}}^{(r)\circ} = \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+}, \mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, \bar{F}))^\vee \subset \Theta_{L/K, \bar{F}}^{(r)}$$

where Tor is defined with respect to the morphism $\mathcal{O}_L \rightarrow \bar{F}$ induced by $i_0: L \rightarrow \bar{K}$. If E is a purely inseparable extension of F , we have $\Theta_{L/K, \bar{F}}^{(r)\circ} = \Theta_{L/K, \bar{F}}^{(r)}$.

The canonical injection $\mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, E) \rightarrow N_{T/Q} \otimes_{\mathcal{O}_L} E$ (1.42) induces a surjection

$$(3.30) \quad N_{T/Q, \bar{F}}^{(r)} \rightarrow \Theta_{L/K, \bar{F}}^{(r)}$$

of vector bundles on $T_{\bar{F}}$. If $T \rightarrow Q$ is minimal, by Lemma 1.3.1.2, the morphism (3.30) is an isomorphism.

Lemma 3.3.7. *Let $r > 1$ and assume that the ramification of L over K is bounded by $r+$. There exists a finite étale morphism*

$$(3.31) \quad X_{L/K}^{(r)} \longrightarrow \Theta_{L/K, \bar{F}}^{(r)}$$

endowed with a bijection of finite sets $\bar{T}_{\bar{F}} \rightarrow X_{L/K, \bar{F}}^{(r)} \times_{\Theta_{L/K, \bar{F}}^{(r)}} T_{\bar{F}}$ satisfying the following property: Let $T = \mathrm{Spec} \mathcal{O}_L \rightarrow Q$ be an immersion to smooth schemes Q over $S = \mathrm{Spec} \mathcal{O}_K$. Then, there exists a commutative diagram

$$(3.32) \quad \begin{array}{ccccc} \bar{T}_{\bar{F}} & \longrightarrow & Q_{\bar{F}}^{(r)} & \longrightarrow & Q_{\bar{F}}^{[r]} \\ & \searrow & \downarrow & & \downarrow \\ & & X_{L/K, \bar{F}}^{(r)} & \longrightarrow & \Theta_{L/K, \bar{F}}^{(r)} \end{array}$$

such that the square is cartesian and that the right vertical arrow is the composition of $Q_{\bar{F}}^{[r]} \rightarrow N_{T/Q, \bar{F}}^{(r)}$ (3.12) and $N_{T/Q, \bar{F}}^{(r)} \rightarrow \Theta_{L/K, \bar{F}}^{(r)}$ (3.30).

Proof. If $T \rightarrow Q$ is minimal, the morphism $N_{T/Q, \bar{F}}^{(r)} \rightarrow \Theta_{L/K, \bar{F}}^{(r)}$ (3.30) is an isomorphism and the condition requires that the vertical arrow $Q_{\bar{F}}^{(r)} \rightarrow X_{L/K, \bar{F}}^{(r)}$ in (3.32) is an isomorphism.

Let $T \rightarrow Q_0$ be a minimal immersion to a smooth scheme over S and set $Q_{0, \bar{F}}^{(r)} = X_{L/K, \bar{F}}^{(r)}$. Then by Lemma 1.2.1.2 and Lemma 1.2.3.2, after replacing Q by an étale neighborhood of T , there is a smooth morphism $Q \rightarrow Q_0$ compatible with the immersions of T . Hence we obtain a required commutative diagram (3.32) with cartesian square. \square

Since a morphism of connected finite étale schemes is uniquely determined on the induced mapping on the fibers of a point, the finite étale scheme $X_{L/K, \bar{F}}^{(r)} \rightarrow \Theta_{L/K, \bar{F}}^{(r)}$ (3.31) is characterized uniquely up to a unique isomorphism by the property in Lemma 3.3.7. If we fix a morphism $i_0: L \rightarrow \bar{K}$, similarly as $Q_{\bar{F}}^{(r)\circ} \rightarrow Q_{\bar{F}}^{[r]\circ}$, we define connected components

$$(3.33) \quad X_{L/K, \bar{F}}^{(r)\circ} \rightarrow \Theta_{L/K, \bar{F}}^{(r)\circ}.$$

Since the construction of $X_{L/K, \bar{F}}^{(r)}$ is functorial in L , by sending L to (3.31), we obtain a functor

$$(3.34) \quad \begin{aligned} & \text{(Finite separable extensions of } K \text{ of ramification bounded by } r+) \\ & \rightarrow \text{(Finite étale morphism of smooth schemes over } \bar{F}). \end{aligned}$$

We identify $F^r(L) = \pi_0(X_{L/K, \bar{F}}^{(r)})$ and $F(L) = F^{r+}(L) = X_{L/K, \bar{F}}^{(r)} \times_{\Theta_{L/K, \bar{F}}^{(r)}} T_{\bar{F}}$.

Assume that L is a Galois extension. Then the Galois group G has a natural action on $X_{L/K}^{(r)} \rightarrow \Theta_{L/K, \bar{F}}^{(r)}$ (3.31). By the injection $\text{Tor}_1^{\mathcal{O}_L}(\Omega_{T/S}^1, E) \rightarrow H_1(L_{E/S})$ (1.36), the inertia subgroup $I \subset G$ acts trivially on $\text{Tor}_1^{\mathcal{O}_L}(\Omega_{T/S}^1, E)$ and hence on $\Theta_{L/K, \bar{F}}^{(r)}$.

Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of henselian discrete valuation rings of ramification index $e_{K'/K} = e \geq 1$. Let $r > 1$ be a rational number and set $r' = er$ or $r' = er - m > 1$ as in Lemma 3.3.5. Let $L \rightarrow L'$ be a morphism of finite separable extension of K and of K' . Assume that the ramification of L over K is bounded by $r+$ and that the ramification of L' over K' is by $r'+$. Then, the commutative diagram (3.18) gives a commutative diagram

$$(3.35) \quad \begin{array}{ccc} X_{L'/K', \bar{F}'}^{(r')} & \longrightarrow & \Theta_{L'/K', \bar{F}'}^{(r')} \\ \downarrow & & \downarrow \\ X_{L/K, \bar{F}}^{(r)} & \longrightarrow & \Theta_{L/K, \bar{F}}^{(r)}. \end{array}$$

3.4. Construction of an automorphism. Let Q be a scheme and $T, T' \subset Q$ be closed subschemes. For a point $t \in Q$ and an integer $n \geq 0$, we say that $T \equiv T' \pmod{\mathfrak{m}_t^n}$, if for the morphism $Q_n = \text{Spec } \mathcal{O}_{Q,t}/\mathfrak{m}_t^n \rightarrow Q$ of schemes, we have an equality $T \times_Q Q_n = T' \times_Q Q_n$ of closed subschemes of Q_n .

Lemma 3.4.1. *Let K be a henselian discrete valuation field and L and L' be finite separable extensions. Let Q be a smooth scheme over $S = \text{Spec } \mathcal{O}_K$ and $T = \text{Spec } \mathcal{O}_L \rightarrow Q$ and $T' = \text{Spec } \mathcal{O}_{L'} \rightarrow Q$ be closed immersions. Let $n \geq 2$ be an integer and assume $T \equiv T' \pmod{\mathfrak{m}_t^n}$ for the closed point $t \in T \subset Q$. Let $Q_F = Q \times_S \text{Spec } F \subset Q$ be the closed fiber and let $Q^{[n]} = Q^{[nQ_F \cdot T]}$ and $Q^{[n]'} = Q^{[nQ_F \cdot T']}$ denote the dilations.*

1. *The canonical isomorphism (1.25) induces isomorphisms*

$$(3.36) \quad \begin{aligned} N_{T/Q} \otimes_{\mathcal{O}_T} E &\rightarrow \text{Ker}(\mathfrak{m}_{Q,t}/\mathfrak{m}_{Q,t}^2 \rightarrow \mathfrak{m}_{T,t}/\mathfrak{m}_{T,t}^2) \\ &= \text{Ker}(\mathfrak{m}_{Q,t}/\mathfrak{m}_{Q,t}^2 \rightarrow \mathfrak{m}_{T',t}/\mathfrak{m}_{T',t}^2) \leftarrow N_{T'/Q} \otimes_{\mathcal{O}_{T'}} E. \end{aligned}$$

2. *We have an equality of dilations $Q^{[n]} = Q^{[n]'}$.*

3. *Let $s: T_{\bar{F}} = \text{Spec } (E \otimes_F \bar{F})_{\text{red}} \rightarrow N_{T/Q, \bar{F}}^{(n)}$ denote the section of the vector bundle defined by the morphism $T' \rightarrow Q^{[n]'} = Q^{[n]}$ lifting $T' \rightarrow Q$ and the isomorphism $Q_F^{[n]} \rightarrow N_{T/Q, F}^{(n)}$ (3.12). Let $+s: N_{T/Q, F}^{(n)} \rightarrow N_{T'/Q, F}^{(n)}$ denote the translation by s and let*

$$(3.37) \quad u: N_{T/Q, F}^{(n)} \rightarrow N_{T'/Q, F}^{(n)}$$

be the isomorphism induced by (3.36). Then, the diagram

$$(3.38) \quad \begin{array}{ccc} Q_F^{[n]} & \xrightarrow{(3.12)} & N_{T/Q, F}^{(n)} \\ \parallel & & \downarrow u \circ +s \\ Q_F^{[n]'} & \xrightarrow{(3.12)} & N_{T'/Q, F}^{(n)} \end{array}$$

is commutative.

Proof. 1. The arrows in (3.36) are the isomorphisms in Lemma 1.2.4.1. By the assumption that $\text{Spec } \mathcal{O}_L/\mathfrak{m}_L^n = \text{Spec } \mathcal{O}_{L'}/\mathfrak{m}_{L'}^n$ and $n \geq 2$, we have the equality in (3.36).

2. Also by $\text{Spec } \mathcal{O}_L/\mathfrak{m}_L^n = \text{Spec } \mathcal{O}_{L'}/\mathfrak{m}_{L'}^n$ and $n \geq 2$, we have $Q^{[n]} = Q^{[n]'}$ by Proposition 3.1.3.1.

3. This follows from Corollary 3.1.4. \square

Lemma 3.4.2. *Let K be a henselian discrete valuation field and L and L' be finite separable extensions. Let Q be a smooth scheme over $S = \text{Spec } \mathcal{O}_K$ and $T = \text{Spec } \mathcal{O}_L \rightarrow Q$ and $T' = \text{Spec } \mathcal{O}_{L'} \rightarrow Q$ be closed immersions. Let $n \geq 2$ be an integer such that $T \equiv T' \pmod{\mathfrak{m}_t^n}$ for the closed point $t \in T \subset Q$. Let $Q^{[n]} = Q^{[nQ_F \cdot T]}$ and $Q^{[n]'} = Q^{[nQ_F \cdot T']}$ be the dilations and $u: N_{T/Q, F}^{(n)} \rightarrow N_{T'/Q, F}^{(n)}$ be the isomorphism (3.37) as in Lemma 3.4.1. Let $f: T' \rightarrow T$ be a morphism over S inducing the identity on the residue field E .*

Then, there exist an étale neighborhood $Q' \rightarrow Q$ of T' and a morphism $\tilde{f}: Q' \rightarrow Q$ lifting $f: T' \rightarrow T$ such that under the identification $Q_F^{[n]} = Q'^{[n]Q'_F \cdot T'}$ induced by the étale morphism $Q' \rightarrow Q$, the diagram

$$(3.39) \quad \begin{array}{ccc} Q_F^{[n]'} & \xrightarrow{(3.12)} & N_{T'/Q, F}^{(n)} \\ \tilde{f} \downarrow & & \downarrow u^{-1} \\ Q_F^{[n]} & \xrightarrow{(3.12)} & N_{T/Q, F}^{(n)} \end{array}$$

is commutative.

The commutative diagrams (3.38) and (3.39) define a commutative diagram

$$(3.40) \quad \begin{array}{ccc} Q_{\bar{F}}^{(n)} & \longrightarrow & N_{T/Q, \bar{F}}^{(n)} \\ \downarrow & & \downarrow +s \\ Q_{\bar{F}}^{(n)} & \longrightarrow & N_{T/Q, \bar{F}}^{(n)}. \end{array}$$

Further if $f: T' \rightarrow T$ is an isomorphism, the left vertical arrow is an isomorphism and the diagram (3.40) is cartesian.

Proof. Let $D \subset Q$ be a regular effective Cartier divisor meeting T transversally at the closed point $t = \text{Spec } E$. Since $T \equiv T' \pmod{\mathfrak{m}_t^2}$, the divisor D meets T' also transversally at t . Since $f: T' \rightarrow T$ induces the identity on $t = \text{Spec } E = T \cap D \rightarrow T' \cap D$, the morphism f and the identity 1_D are extended to a morphism $\tilde{f}_1: T' \cup D \rightarrow T \cup D$ of closed subschemes of Q . Since Q is smooth over S , there exists an étale neighborhood $Q' \rightarrow Q$ of t and a morphism $\tilde{f}: Q' \rightarrow Q$ lifting \tilde{f}_1 .

We show the commutativity of the square (3.39). We identify $\mathfrak{m}_{Q,t}/\mathfrak{m}_{Q,t}^2 = \mathfrak{m}_{Q',t}/\mathfrak{m}_{Q',t}^2$ and $\mathfrak{m}_{T,t}/\mathfrak{m}_{T,t}^2 = \mathfrak{m}_{T',t}/\mathfrak{m}_{T',t}^2$. Then by (3.36), it suffices to show that $\tilde{f}: Q' \rightarrow Q$ induces the identity on $\text{Ker}(\mathfrak{m}_{Q,t}/\mathfrak{m}_{Q,t}^2 \rightarrow \mathfrak{m}_{T,t}/\mathfrak{m}_{T,t}^2)$. This follows from the isomorphism (1.26) and the fact that \tilde{f}_1 induces the identity on D and hence on $N_{t/D}$.

Combining (3.38) and (3.39) together with the functoriality of (3.9), we obtain a commutative diagram

$$\begin{array}{ccccc} Q_{\bar{F}}^{(n)} & \xrightarrow{(3.9)} & Q_{\bar{F}}^{[n]} & \xrightarrow{(3.12)} & N_{T/Q, \bar{F}}^{(n)} \\ \parallel & & \parallel & & \downarrow u \circ +s \\ Q'_{\bar{F}}^{(n)} & \xrightarrow{(3.9)} & Q'_{\bar{F}}^{[n]} & \xrightarrow{(3.12)} & N_{T'/Q, \bar{F}}^{(n)} \\ \tilde{f} \downarrow & & \tilde{f} \downarrow & & \downarrow u^{-1} \\ Q_{\bar{F}}^{(n)} & \xrightarrow{(3.9)} & Q_{\bar{F}}^{[n]} & \xrightarrow{(3.12)} & N_{T/Q, \bar{F}}^{(n)}. \end{array}$$

Thus, we obtain (3.40).

By the assumption that T meets D transversally at t , the canonical morphism $\mathfrak{m}_{Q,t}/\mathfrak{m}_{Q,t}^2 \rightarrow \mathfrak{m}_{T,t}/\mathfrak{m}_{T,t}^2 \times \mathfrak{m}_{D,t}/\mathfrak{m}_{D,t}^2$ is an isomorphism. Similarly, $\mathfrak{m}_{Q',t}/\mathfrak{m}_{Q',t}^2 \rightarrow \mathfrak{m}_{T',t}/\mathfrak{m}_{T',t}^2 \times \mathfrak{m}_{D,t}/\mathfrak{m}_{D,t}^2$ is an isomorphism. Hence if $f: T' \rightarrow T$ is an isomorphism, then $\tilde{f}: Q' \rightarrow Q$ is étale. This implies that the left vertical arrow of (3.40) is an isomorphism and the diagram itself is cartesian. \square

Let $f, g: S = \text{Spec } A \rightarrow X$ be morphisms of schemes and $I \subset A$ be an ideal. If the restrictions of f and g to the closed subscheme $\text{Spec } A/I \subset S$ are the same, we write $f \equiv g \pmod{I}$.

Lemma 3.4.3. *Let K be a henselian discrete valuation field and let L be a finite separable extension of K . Let*

$$(3.41) \quad \begin{array}{ccc} Q & \longleftarrow & T = \text{Spec } \mathcal{O}_L \\ \downarrow & & \downarrow \\ P & \xleftarrow{i} & S = \text{Spec } \mathcal{O}_K \end{array}$$

be a cartesian diagram of separated schemes over S such that the horizontal arrows are closed immersions to smooth schemes over S and that the vertical arrows are finite and flat. Let $t \in T \subset Q$ be the closed point and let $n \geq 2$ be an integer.

1. *Let $i': S \rightarrow P$ be a closed immersion over S such that $i' \equiv i \pmod{\mathfrak{m}_K^n}$. Let $T' = Q \times_P S$ be the fiber product with respect to $i': S \rightarrow P$ and set $T' = \text{Spec } A$.*

Then, we have $T \equiv T' \pmod{\mathfrak{m}_t^n}$ and A is a discrete valuation ring.

2. *Let $P^{[n]} = P^{[n]P \cdot S}$ denote the dilatation and let $S_1 = \text{Spec } \mathcal{O}_{K_1} \rightarrow P^{[n]}$ be the henselization at the generic point ξ of the closed fiber $P_F^{[n]} = P^{[n]} \times_S \text{Spec } F$. Let $i_1, i'_1: S_1 \rightarrow P_1 = P \times_S S_1$ be the sections induced by the compositions $S_1 \rightarrow S \xrightarrow{i} P$ and by $S_1 \rightarrow P^{[n]} \rightarrow P$ respectively and consider the cartesian diagram*

$$(3.42) \quad \begin{array}{ccccc} T_1 & \longrightarrow & Q_1 = Q \times_S S_1 & \longleftarrow & T'_1 \\ \downarrow & & \downarrow & & \downarrow \\ S_1 & \xrightarrow{i_1} & P_1 = P \times_S S_1 & \xleftarrow{i'_1} & S_1. \end{array}$$

Let $t_1 \in T_1 \subset Q_1$ denote the closed point.

Then, \mathcal{O}_{K_1} is a discrete valuation ring and the residue field $F_1 = k(\xi)$ is a purely transcendental extension of F . We have $i_1 \equiv i'_1 \pmod{\mathfrak{m}_{K_1}^n}$ and $T_1 \equiv T'_1 \pmod{\mathfrak{m}_{t_1}^n}$ and the schemes T_1 and T'_1 are spectrums of discrete valuation rings.

Proof. 1. We consider S as a closed subscheme of P by the section i and let $s \in S \subset P$ be the closed point. Let $S' \subset P$ denote the closed subscheme defined by the section $i': S \rightarrow P$. Then, since $i \equiv i' \pmod{\mathfrak{m}_K^n}$, we have $S \equiv S' \pmod{\mathfrak{m}_s^n}$. Hence we have $T \equiv T' \pmod{\mathfrak{m}_t^n}$.

By $T \equiv T' \pmod{\mathfrak{m}_t^n}$, the maximal ideal of A/\mathfrak{m}_A^n is generated by one element. Since $n \geq 2$, by Nakayama's lemma, \mathfrak{m}_A is generated by one

element. Since A is finite flat over \mathcal{O}_K , we have $\dim A = 1$ and hence A is a discrete valuation ring.

2. By Lemma 3.1.2.2, $P^{[n]}$ is smooth over S and the closed fiber $P_F^{[n]}$ is a vector space over F . Hence \mathcal{O}_{K_1} is a discrete valuation ring and the residue field $F_1 = k(\xi)$ is a purely transcendental extension of F . By the definition of dilatations, the morphism $P^{[n]} \rightarrow P$ is congruent to the composition $P^{[n]} \rightarrow P \rightarrow S \xrightarrow{i} P$ modulo the pull-back of \mathfrak{m}_K^n . Hence, we have $i_1 \equiv i'_1 \pmod{\mathfrak{m}_{K_1}^n}$.

The fiber product $T_1 = Q_1 \times_{P_1} S_1$ is equal to $T \times_S S_1$ with respect to the composition $S_1 \rightarrow P^{[n]} \rightarrow P \rightarrow S$. Since $P^{[n]} \rightarrow S$ is smooth and the residue field $F_1 = k(\xi)$ is a purely transcendental extension of F , we have $T_1 = T \times_S S_1 = \text{Spec } \mathcal{O}_{L_1}$ for a discrete valuation ring \mathcal{O}_{L_1} . By 1. applied to $i_1 \equiv i'_1 \pmod{\mathfrak{m}_{K_1}^n}$, we have $T_1 \equiv T'_1 \pmod{\mathfrak{m}_{L_1}^n}$ and $T'_1 = \text{Spec } \mathcal{O}_{L'_1}$ for a discrete valuation ring $\mathcal{O}_{L'_1}$. \square

Lemma 3.4.4. *Let K be a henselian discrete valuation field and let L be a finite separable extension of K . Let $n \geq 2$ be an integer and cartesian diagrams (3.41) and (3.42) be as in Lemma 3.4.3. Assume that the ramification of L over K is bounded by $n+$.*

1. *There exists an open neighborhood $W \subset P^{[n]}$ of the closed fiber $P_F^{[n]} \subset P^{[n]}$ such that on the generic fiber $W_K = W \times_S \text{Spec } K$, the base change $Q_{W_K} = Q \times_P W_K \rightarrow W_K$ is étale.*

2. *There exists a finite separable extension L'_1 of K_1 such that $T'_1 = \text{Spec } \mathcal{O}_{L'_1}$.*

3. *Let $T_{W_K} = T \times_S W_K$ and define a cartesian diagram*

$$(3.43) \quad \begin{array}{ccc} A & \longleftarrow & A_K = \text{Isom}_{W_K}(Q_{W_K}, T_{W_K}) \\ \downarrow & & \downarrow \\ W & \longleftarrow & W_K \end{array}$$

by letting A denote the normalization of W in A_K . Let $A^{\text{ét}} \subset A$ denote the largest open subscheme étale over W .

Then, the intersection $\text{Im}(A^{\text{ét}} \rightarrow W) \cap P_F^{[n]}$ is dense in the closed fiber $P_F^{[n]}$. Further, the universal isomorphism $Q \times_P A_K \rightarrow T \times_S A_K$ is extended to an isomorphism $Q \times_P A^{\text{ét}} \rightarrow T \times_S A^{\text{ét}}$.

Proof. 1. By the reduced fiber theorem [6], there exists a finite separable extension K' of ramification index e such that the morphism $Q_{S'}^{(en)} \rightarrow S' = \text{Spec } \mathcal{O}_{K'}$ has reduced geometric fibers. By the assumption that the ramification of L over K is bounded by $n+$ and by Proposition 3.3.6 (1) \Rightarrow (4), the finite morphism $Q_{S'}^{(en)} \rightarrow P_{S'}^{[en]}$ is étale on a neighborhood W' of the closed fiber $P_{F'}^{[en]}$. Since $S' \rightarrow S$ is faithfully flat, the image $W \subset P^{[n]}$ of W' satisfies the required condition.

2. Let $i_1: S_1 \rightarrow P_1 = P \times_S S_1$ denote the section induced by $S_1 \rightarrow S \xrightarrow{i} P$ and let $i'_1: S_1 \rightarrow P_1$ denote the section induced by $S_1 \rightarrow P^{[n]} \rightarrow P$ as in Lemma 3.4.3. Then, by Lemma 3.4.3.2, we have $T' = \text{Spec } \mathcal{O}_{L'_1}$ for a finite morphism $\mathcal{O}_{K_1} \rightarrow \mathcal{O}_{L'_1}$ of discrete valuation rings.

Since $\xi \in P_F^{[n]} \subset W$, the canonical morphism $S_1 \rightarrow P^{[n]}$ factors through $S_1 \rightarrow W$. Since $\text{Spec } L'_1 = Q_{W_K} \times_{W_K} \text{Spec } K_1$, the finite extension L'_1 of K_1 is separable.

3. We apply Proposition 3.2.2 to $W \supset W_K \leftarrow A_K$. Similarly as in the proof of 2., the lifting $S \rightarrow P^{[n]}$ of $i: S \rightarrow P$ factors through $S \rightarrow W$. The cartesian diagram (3.41) induces an isomorphism $Q_{W_K} \times_{W_K} \text{Spec } K \rightarrow T_{W_K} \times_{W_K} \text{Spec } K$. This defines a section of the finite étale scheme $A_K \rightarrow W_K$ at the point $\text{Spec } K \subset S \rightarrow P^{[n]}$. By the universality of the normalization, the section $\text{Spec } K \rightarrow A_K$ is extended to a section $S \rightarrow A$.

Let S' be as in the proof of 1. Changing W' if necessary, we may assume that the finite morphism $Q_{S'}^{(en)} \rightarrow P_{S'}^{[en]}$ is étale on $W' = W \times_S S'$. We may further assume that K' contains a Galois closure of L . Then the normalization $\bar{T}_{S'} \rightarrow S'$ of $T \times_S S'$ is a split finite étale covering. Hence $A' = \text{Isom}_{W'}(Q_{S'}^{(en)} \times_{P_{S'}^{(en)}} W', \bar{T}_{S'} \times_{S'} W')$ is a finite étale scheme over W' .

Since $A' \rightarrow S'$ is the normalization of $W' \subset P^{[n]} \times_S S'$ in $A_K \times_{W_K} W'_{K'}$, it follows that $A^{\text{ét}}$ is an open neighborhood of the section $S \rightarrow W$ by Proposition 3.2.2. Since $P_F^{[n]}$ is irreducible, the non-empty intersection $\text{Im}(A^{\text{ét}} \rightarrow W) \cap P_F^{[n]}$ is dense.

Since $A^{\text{ét}} \rightarrow W$ is étale, $Q \times_P A^{\text{ét}}$ and $T \times_S A^{\text{ét}}$ are the normalizations of $A^{\text{ét}}$ in $Q \times_P A_K$ and $T \times_S A_K$. Hence the universal isomorphism $Q \times_P A_K \rightarrow T \times_S A_K$ is extended to an isomorphism $Q \times_P A^{\text{ét}} \rightarrow T \times_S A^{\text{ét}}$. \square

We prepare a crucial construction in the proof of additivity Theorem 4.3.3.

Proposition 3.4.5. *Let K be a henselian discrete valuation field and let L be a finite separable extension of K . Let*

$$(3.44) \quad \begin{array}{ccc} Q & \longleftarrow & T = \text{Spec } \mathcal{O}_L \\ \downarrow & & \downarrow \\ P & \xleftarrow{i} & S = \text{Spec } \mathcal{O}_K \end{array}$$

be a cartesian diagram of schemes over S such that the horizontal arrows are closed immersions to smooth schemes over S and that the vertical arrows are finite and flat. Let $n \geq 2$ be an integer such that the ramification of L over K is bounded by $n+$.

Assume that the residue field E of L is a purely inseparable extension of the residue field F of K . Let $F_1 = k(\xi)$ be the residue field at the generic point $\xi \in P_F^{(n)} = P^{(n)} \times_S \text{Spec } F$ of the closed fiber and $\bar{F} \rightarrow \bar{F}_1$ be a morphism of algebraic closures. Then, there exists a cartesian diagram

$$(3.45) \quad \begin{array}{ccc} Q_{\bar{F}}^{(n)} \times_{\bar{F}} \bar{F}_1 & \longrightarrow & N_{T/Q, \bar{F}}^{(n)} \times_{\bar{F}} \bar{F}_1 \\ \downarrow & & \downarrow +\xi \\ Q_{\bar{F}}^{(n)} \times_{\bar{F}} \bar{F}_1 & \longrightarrow & N_{T/Q, \bar{F}}^{(n)} \times_{\bar{F}} \bar{F}_1 \end{array}$$

where the horizontal arrows are the base change of the composition of (3.9) and (3.12).

Proof. We use the notation in (3.42) in Lemma 3.4.3. We identify F_1 with the residue field at the closed point of S_1 . The horizontal arrow $Q_{\bar{F}}^{(n)} \times_{\bar{F}} \bar{F}_1 \rightarrow N_{T/Q, \bar{F}}^{(n)} \times_{\bar{F}} \bar{F}_1$ (3.45) is canonically identified with $Q_{1, \bar{F}_1}^{(n)} \rightarrow N_{T_1/Q_1, \bar{F}_1}^{(n)}$ defined for $T_1 \rightarrow Q_1$ over S_1 in (3.42).

The finite extension L'_1 is a separable extension of K_1 by Lemma 3.4.4.2. Hence by Lemma 3.4.2, in order to define a cartesian diagram (3.45), it suffices to define a finite unramified extension K_2 of K_1 and an isomorphism $\mathcal{O}_{L_2} \rightarrow \mathcal{O}_{L'_2}$ for $L_2 = L_1 K_2$ and $L'_2 = L'_1 K_2$ inducing the identity on the residue fields E_2 of \mathcal{O}_{L_2} . To construct them, we apply Lemma 3.4.4.3.

As in Lemma 3.4.4.3, let A be the normalization of $W \subset P^{[n]}$ in $A_K = \text{Isom}_{W_K}(Q_{W_K}, T_{W_K})$ over the generic fiber $W_K \subset W$ and let $A^{\text{ét}} \subset A$ denote the largest open subscheme étale over W . By Lemma 3.4.4.3, the intersection $\text{Im}(A^{\text{ét}} \rightarrow W) \cap P_F^{[n]}$ is dense in $P_F^{[n]}$. Take a point $\xi_2 \in A^{\text{ét}}$ above the generic point $\xi \in P_F^{[n]}$ and let $S_2 = \text{Spec } \mathcal{O}_{K_2}$ be the henselization of A at ξ_2 . Then K_2 is a finite unramified extension of K_1 .

Since $T_1 = T \times_S S_1$ and $T'_1 = Q \times_P S_1$, by pulling back the universal isomorphism $Q \times_P A^{\text{ét}} \rightarrow T \times_S A^{\text{ét}}$ on $A^{\text{ét}}$ by $S_2 \rightarrow A^{\text{ét}}$, we obtain an isomorphism $f: T'_2 = T'_1 \times_{S_1} S_2 \rightarrow T_2 = T \times_S S_2$. Since E is assumed to be purely inseparable over F , the residue field $E_2 = EF_2$ of T_2 is also purely inseparable over the residue field F_2 of K_2 and hence the morphism f induces the identity on E_2 . Thus by applying Lemma 3.4.2 to the isomorphism f , we obtain a cartesian diagram (3.45). \square

4. RAMIFICATION GROUPS

4.1. Ramification groups.

Theorem 4.1.1. ([1], [18]) *Let K be a henselian discrete valuation field. Let L be a finite Galois extension of K of Galois group G .*

1. ([1, Theorems 3.3], [18, Theorem 3.3.1 (1)]) *There exists a unique decreasing filtration (G^r) indexed by rational numbers $r > 0$ by normal subgroups of G indexed by positive rational numbers such that, for every rational number $r > 0$ and for every intermediate extension $K \subset M \subset L$, the morphism $F \rightarrow F^r$ of functors induces a bijection*

$$(4.1) \quad F(M)/G^r \rightarrow F^r(M)$$

of finite G_K -sets.

2. ([1, Theorems 3.8], [18, Theorem 3.3.1 (2)]) *There exists a finite increasing sequence $0 = r_0 < r_1 < \dots < r_n$ of rational numbers such that G^r is constant in the semi-open intervals $(r_{i-1}, r_i]$ for $i = 1, \dots, n$ and in (r_n, ∞) .*

3. ([18, Theorem 3.3.1 (1)]) *For a rational number $r > 0$, set $G^{r+} = \bigcup_{s>r} G^s$. Then, for every rational number $r > 0$ and for every intermediate extension $K \subset M \subset L$, the morphism $F \rightarrow F^{r+}$ of functors induces a bijection*

$$(4.2) \quad F(M)/G^{r+} \rightarrow F^{r+}(M)$$

of finite G_K -sets.

4. ([1, Proposition 3.7], [18, Proposition 3.3.5]) *The subgroups $G^1 \supset G^{1+}$ equal the inertia subgroup I and its p -Sylow subgroup P .*

Corollary 4.1.2. *Let L be a finite Galois extension of K . Let $r > 1$ be a rational number and assume that the ramification of L over K is bounded by $r+$. Let $T = \text{Spec } \mathcal{O}_L \rightarrow \mathcal{O}_K$ be an immersion to a smooth scheme over $S = \text{Spec } \mathcal{O}_K$. Fix a morphism $i_0: L \rightarrow \bar{K}$ and let $Q_{\bar{F}}^{(r)\circ} \subset Q_{\bar{F}}^{(r)}$ and $Q_{\bar{F}}^{[r]\circ} \subset Q_{\bar{F}}^{[r]}$ (3.13) denote the connected components.*

1. $Q_{\bar{F}}^{(r)\circ}$ is a G^r -torsor over $Q_{\bar{F}}^{[r]\circ}$ and $X_{L/K, \bar{F}}^{(r)\circ}$ (3.33) is a G^r -torsor over $\Theta_{L/K, \bar{F}}^{(r)\circ}$.
2. The G^r -torsor $Q_{\bar{F}}^{(r)\circ}$ over $Q_{\bar{F}}^{[r]\circ}$ is additive if and only if the G^r -torsor $X_{L/K, \bar{F}}^{(r)\circ}$ over $\Theta_{L/K, \bar{F}}^{(r)\circ}$ is additive.

Proof. 1. Since $r > 1$, the subgroup $G^r \subset G^{1+} = P$ acts trivially on $\Theta_{L/K, \bar{F}}^{(r)}$ and acts on $X_{L/K, \bar{F}}^{(r)\circ}$ as an automorphism over $\Theta_{L/K, \bar{F}}^{(r)\circ}$. Since the fiber $F^{r+}(L) \cap X_{L/K, \bar{F}}^{(r)\circ} \subset F^{r+}(L) = F(L)$ of $X_{L/K, \bar{F}}^{(r)\circ} \rightarrow \Theta_{L/K, \bar{F}}^{(r)\circ}$ at 0 is a G^r -torsor, the finite étale scheme $X_{L/K, \bar{F}}^{(r)\circ}$ over $\Theta_{L/K, \bar{F}}^{(r)\circ}$ is a G^r -torsor. By the cartesian square in (3.32), $Q_{\bar{F}}^{(r)\circ}$ is also a G^r -torsor over $Q_{\bar{F}}^{[r]\circ}$.

2. This follows from the cartesian square in (3.32) and Corollary 2.1.8.3. □

The filtrations $(G^r)_{r>0}$ and $(G^{r+})_{r>0}$ satisfy the following functoriality. Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of henselian discrete valuation field of ramification index e . Let $L \rightarrow L'$ be a morphism of finite Galois extensions of K and of K' . Then, for rational numbers $r > 0$ and $r' = er$ or $r' = er - m > 0$ as in Proposition 3.3.2, the commutative diagram (3.26) implies that the morphism $G' = \text{Gal}(L'/K') \rightarrow G = \text{Gal}(L/K)$ induces mappings $G'/G'^{r'+} \rightarrow G/G^{r+}$ and $G'/G'^{r'} \rightarrow G/G^r$ and hence morphisms

$$(4.3) \quad G'^{r'+} \rightarrow G^{r+}, \quad G'^{r'} \rightarrow G^r.$$

We define the total dimension of a Galois representation. Let C be a field of characteristic different from p and let V be a representation of G on a C -vector space of finite dimension. Then, since G^{1+} is a p -group, there exists a unique decomposition

$$(4.4) \quad V = \bigoplus_{r \in \mathbf{Q}, r \geq 1} V^{(r)}$$

such that for every rational number $r > 1$ and for the G^r -fixed part, we have

$$(4.5) \quad V^{G^r} = \bigoplus_{1 \leq s < r} V^{(s)}.$$

The decomposition (4.4) is called the slope decomposition. The total dimension is defined by

$$(4.6) \quad \text{dimtot}(V) = \sum_{r \in \mathbf{Q}, r \geq 1} r \cdot \dim V^{(r)}$$

as a rational number $\geq \dim V$. The equality is equivalent to the condition that the action of $P = G^{1+}$ on V is trivial. In the classical case where the residue field is perfect, the total dimension is the sum of the dimension and the Swan conductor: $\dim_{\text{tot}}(V) = \dim V + \text{Sw } V$ and is known to be an integer by [22, Théorème 1', Section 2, Chapitre VI].

4.2. Reduction steps. We recall some facts on discrete valuation rings.

Lemma 4.2.1 ([4, Lemme 2.2.1]). *Let K be a henselian discrete valuation field and \hat{K} be the completion. Then, the completion defines an equivalence of categories*

(4.7)

(Finite separable extensions of K) \rightarrow (Finite separable extensions of \hat{K}).

For a finite separable extension L over K , the canonical morphism

$$(4.8) \quad \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{\hat{K}} \rightarrow \mathcal{O}_{\hat{L}}$$

is an isomorphism.

Proposition 4.2.2. *Let K_0 be a henselian discrete valuation field and let F be a separable extension of the residue field k . Then, there exists an inductive system $(\mathcal{O}_\lambda)_{\lambda \in \Lambda}$ of essentially smooth extensions of henselian discrete valuation rings of \mathcal{O}_{K_0} such that the residue field of $\varinjlim_{\lambda \in \Lambda} \mathcal{O}_\lambda$ is F .*

Proof. The construction is similar to that in [9, (10.3.1) Chapitre 0]. Let $(x_i)_{i \in I}$ be a transcendental basis of F over k and let $E_0 \subset F$ be the separable closure of the subfield $F_I = k(x_i; i \in I)$. If k is of characteristic $p > 0$, for integers $n \geq 1$, let $E_n = \{x \in F \mid x^{p^n} \in E_0\} \subset F$ be the subfields so that we have $F = \bigcup_n E_n$.

Let K_I be the henselization \mathcal{O}_{K_I} of the localization of the polynomial ring $\mathcal{O}_{K_0}[x_i; i \in I]$ at the prime ideal $\mathfrak{m}_{K_0} \mathcal{O}_{K_0}[x_i; i \in I]$. Then, the residue field of K_I is F_I . Let L_0 be a unramified extension of K_I with residue field $E_0 \supset F_I$. For each $n \geq 1$, let $(u_{n,i})_{i \in I_n}$ be a family of elements of E_{n-1} such that $du_{n,i} \in \Omega_{E_{n-1}}^1$, $i \in I_n$ is linearly independent and that E_n is generated by the p -th roots of $u_{n,i}$, $i \in I_n$ over E_{n-1} . Take inductively liftings of $u_{n,i}$, $i \in I_n$ in $\mathcal{O}_{L_{n-1}}$ denoted abusively also by $u_{n,i}$, $i \in I_n$ and define $\mathcal{O}_{L_n} = \mathcal{O}_{L_{n-1}}[X_i, i \in I_n]/(X_i^p - u_{n,i}, i \in I_n)$. Then $(\mathcal{O}_{L_n})_{n \in \mathbb{N}}$ is an inductive system of henselian discrete valuation rings and the residue field of $K = \varinjlim_n L_n$ is $F = \varinjlim_n E_n$.

For a finite subset $J \subset I$, let $E_J = k(x_i, i \in J) \subset E_I$ and let $L_J \subset L_I$ be the corresponding subextension. For a finite separable extension $E'_0 \subset E_0$ of E_J , let $L'_0 \subset L_0$ be the corresponding unramified extension. For a sequence $J_j \subset I_j$, $j = 1, \dots, m$ of finite subsets, if subextensions $E'_n = E'_{n-1}(u_{n,i}^{1/p}, i \in J_n)$ for $n = 1, \dots, m$ are defined inductively, let L'_n denote the corresponding extensions. Then, K is the inductive limit of these L'_m .

By the construction, for each L'_m , there exist a flat ring A of finite type over \mathcal{O}_{K_0} , a prime ideal \mathfrak{p} of A and an isomorphism $A_{\mathfrak{p}}^h \rightarrow \mathcal{O}_{L'_m}$. Since the residue field E'_m is a subfield of a separable extension F of k , the extension E'_m itself is a separable extension of k and hence A is smooth over \mathcal{O}_{K_0} at \mathfrak{p} . Namely $\mathcal{O}_{L'_m}$ is essentially smooth over \mathcal{O}_{K_0} as required. \square

Corollary 4.2.3. *Let K be a complete discrete valuation field of characteristic $p > 0$ with residue field F . Let $k \subset F$ be a perfect subfield.*

1. *Assume that K is of characteristic 0. Let $\mathcal{O}_{K_0} = W(k) \subset \mathcal{O}_K$ be the ring of Witt vectors and let $K_0 \rightarrow K$ be a morphism of discrete valuation fields. Then, there exist an inductive system $(\mathcal{O}_{K_\lambda})_{\lambda \in \Lambda}$ of discrete valuation rings essentially of finite type over \mathcal{O}_{K_0} and an isomorphism from the completion of $\varinjlim_{\lambda \in \Lambda} \mathcal{O}_{K_\lambda}$ to \mathcal{O}_K over \mathcal{O}_{K_0} .*

2. *Assume that K is of characteristic $p > 0$ and let $t \in K$ be a uniformizer. Then, there exists an isomorphism $F((t)) \rightarrow K$ of complete discrete valuation fields. Further, there exist an inductive system $(\mathcal{O}_{K_\lambda})_{\lambda \in \Lambda}$ of discrete valuation rings essentially smooth over the henselization $\mathcal{O}_{K_0} = k[t]_{(t)}^h$ of the local ring of the affine line \mathbf{A}_k^1 at the origin and an isomorphism from the completion of $\varinjlim_{\lambda \in \Lambda} \mathcal{O}_{K_\lambda}$ to \mathcal{O}_K over \mathcal{O}_{K_0} .*

Proof. 1. Let $\mathcal{O}_{K_1} = \varinjlim_{\lambda \in \Lambda} A_\lambda$ be a formally smooth extension of K_0 with residue field F and $(A_\lambda)_{\lambda \in \Lambda}$ be an inductive system of henselian discrete valuation rings essentially smooth over \mathcal{O}_{K_0} as in Proposition 4.2.2. Since K is a finite separable extension of the completion \widehat{K}_1 , there exist a finite separable extension L_1 of K_1 and an isomorphism $\widehat{L}_1 \rightarrow K$ by the equivalence (4.7) of categories. Since \mathcal{O}_{L_1} is of finite presentation over \mathcal{O}_{K_1} , there exist $\lambda \in \Lambda$, a finite separable extension K_λ of the fraction field of A_λ and an isomorphism $\mathcal{O}_{K_\lambda} \otimes_{A_\lambda} \mathcal{O}_{K_1} \rightarrow \mathcal{O}_K$. Hence the assertion follows.

2. Since F is formally smooth over k , there exists a morphism $F \rightarrow \mathcal{O}_K$ such that the composition with $\mathcal{O}_K \rightarrow F$ is an isomorphism. This induces an isomorphism $F((t)) \rightarrow K$.

Let $\mathcal{O}_{K_1} = \varinjlim_{\lambda \in \Lambda} \mathcal{O}_{K_\lambda}$ be a formally smooth extension of K_0 with residue field F and $(\mathcal{O}_{K_\lambda})_{\lambda \in \Lambda}$ be an inductive system of henselian discrete valuation rings essentially smooth over \mathcal{O}_{K_0} as in Proposition 4.2.2. Then, since \mathcal{O}_{K_1} is formally smooth over \mathcal{O}_{K_0} and K is complete, we have a morphism $K_1 \rightarrow K$ over K_0 inducing the identity $F \rightarrow F$ and hence an isomorphism $\widehat{K}_1 \rightarrow K$. \square

Lemma 4.2.4. *Let $K \subset K'$ be an extension of henselian discrete valuation fields. Let L be a finite separable extension of K and set $L' = L \otimes_K K'$. Assume that the canonical morphism $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow \mathcal{O}_{L'}$ is an isomorphism. Then, for any intermediate extension $K \subset M \subset L$ and $M' = M \otimes_K K'$, the canonical morphism $\mathcal{O}_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow \mathcal{O}_{M'}$ is an isomorphism.*

Proof. Since the inclusion $\mathcal{O}_M \rightarrow \mathcal{O}_L$ is faithfully flat, the induced morphism $\mathcal{O}_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ is also faithfully flat. Since $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} = \mathcal{O}_{L'}$ is regular, $\mathcal{O}_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ is also regular by [10, Chapitre 0, Proposition (17.3.3) (i)]. \square

We prepare reduction steps of the proof of the main results.

Proposition 4.2.5. *Let $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ be a morphism of henselian discrete valuation rings of ramification index $e \geq 1$. Let L be a finite Galois extension of K of Galois group G . Let $L' = LK'$ be a composition field and let $G' = \text{Gal}(L'/K')$ be the Galois group. Let $r > 1$ and $r' > 1$ be rational numbers and assume one of the following conditions (1)–(3):*

(1) $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ is dominant on the tangent spaces and $r' = r$.

(2) The discrete valuation rings \mathcal{O}_K and $\mathcal{O}_{K'}$ are essentially of finite type and generically separable over a discrete valuation ring \mathcal{O}_{K_0} with perfect residue field k . Integers $m \geq 0$, $q \geq 2$ and $d = \dim_F \Omega_F^1$ and $d' = \dim_{F'} \Omega_{F'}^1$ satisfy the condition (2) in Proposition 1.2.10 and (2) in Proposition 1.3.5.4 and $r \geq \frac{m}{e} \frac{q}{q-1}$ and $r' = er - m \geq \frac{m}{q-1}$.

(3) The morphism $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow \mathcal{O}_{L'}$ is an isomorphism and $r' = er$. Fix compatible morphisms $i_0: L \rightarrow \bar{K}$ and $i'_0: L' \rightarrow \bar{K}'$ to separable closures. Let $X_{L/K, \bar{F}}^{(r)\circ} \subset X_{L/K, \bar{F}}^{(r)}$ and $\Theta_{L/K, \bar{F}}^{(r)\circ} \subset \Theta_{L/K, \bar{F}}^{(r)}$ denote the connected components containing the images of $i_0 \in F(L) \rightarrow F^{r+}(L) \subset X_{L/K, \bar{F}}^{(r)} \rightarrow \Theta_{L/K, \bar{F}}^{(r)}$ and similarly for L'/K' . Assume that the ramification of L is bounded by $r+$.

1. The injection $G' = \text{Gal}(L'/K') \rightarrow G$ induces an isomorphism $G'^{r'} \rightarrow G^r$.

2. In the cases (2) and (3), the G^r -torsor $X_{L/K, \bar{F}}^{(r)\circ}$ over $\Theta_{L/K, \bar{F}}^{(r)\circ}$ is additive if and only if the $G'^{r'}$ -torsor $X_{L'/K', \bar{F}'}^{(r')\circ}$ over $\Theta_{L'/K', \bar{F}'}^{(r')\circ}$ is additive.

Proof. Let

$$(4.9) \quad \begin{array}{ccc} \text{Spec } \mathcal{O}_{L'} = T' & \longrightarrow & Q' \\ & \downarrow & \downarrow \\ \text{Spec } \mathcal{O}_L = T & \longrightarrow & Q \end{array}$$

be a commutative diagram of schemes where the horizontal arrows are closed immersions to smooth schemes over S and over S' and the vertical arrows are compatible with $S' \rightarrow S$. In the case (1), we assume that $T \rightarrow Q$ and $T' \rightarrow Q'$ are minimal. In the case (2), we assume that (4.9) is induced by an essentially cartesian diagram

$$(4.10) \quad \begin{array}{ccc} T' & \longrightarrow & Q'_0 \\ \downarrow & & \downarrow \\ T & \longrightarrow & Q_0 \end{array}$$

of divisorial essential immersions to smooth separated schemes over $S_0 = \text{Spec } \mathcal{O}_{K_0}$. In the case (3), we assume that the diagram (4.9) is cartesian.

By Lemma 3.3.5, the ramification of L' is bounded by $r'+$ and the injection $G' \rightarrow G$ induces an injection $G'^{r'} \rightarrow G^r$. Let

$$(4.11) \quad \begin{array}{ccccc} \bar{T}'_{\bar{F}'} & \longrightarrow & Q'^{(r')}_{\bar{F}'} & \longrightarrow & Q'^{[r']}_{\bar{F}'} \\ \downarrow & & \downarrow & & \downarrow \\ \bar{T}_{\bar{F}} & \longrightarrow & Q^{(r)}_{\bar{F}} & \longrightarrow & Q^{[r]}_{\bar{F}} \end{array}$$

be the commutative diagram induced by (4.9). Since $i'_0: L' \rightarrow \bar{K}'$ and $i_0: L \rightarrow K$ are compatible, we obtain a commutative diagram

$$(4.12) \quad \begin{array}{ccc} Q_{\bar{F}'}^{l(r')\circ} & \longrightarrow & Q_{\bar{F}'}^{l[r']\circ} \\ \downarrow & & \downarrow \\ Q_{\bar{F}}^{(r)\circ} & \longrightarrow & Q_{\bar{F}}^{[r]\circ} \end{array}$$

of the connected components. The horizontal arrows are a G^r -torsor and a $G'^{r'}$ -torsor respectively and the left vertical arrow compatible with $G'^{r'} \rightarrow G^r$.

In the case (1), since $T \rightarrow Q$ and $T' \rightarrow Q'$ are assumed minimal, the right vertical arrow of (4.12) is identified with

$$\Theta_{L'/K', \bar{F}'}^{(r')\circ} = \text{Spec } S(\text{Tor}_1^{\mathcal{O}_{L'}}(\Omega_{\mathcal{O}_{L'}/\mathcal{O}_{K'}}^1, \bar{F}')) \rightarrow \Theta_{L/K, \bar{F}}^{(r)\circ} = \text{Spec } S(\text{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, \bar{F})).$$

Since the morphism $S' \rightarrow S$ is assumed to be dominant on the tangent spaces at the closed point $s' \in S'$, the morphism $S(H_1(L_{\bar{F}/S})) \rightarrow S(H_1(L_{\bar{F}'/S'}))$ of symmetric algebras is an injection. By Lemma 1.3.1.1, the morphism $S(\text{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, \bar{F})) \rightarrow S(\text{Tor}_1^{\mathcal{O}_{L'}}(\Omega_{\mathcal{O}_{L'}/\mathcal{O}_{K'}}^1, \bar{F}'))$ is also an injection. Hence, the right vertical arrow of (4.12) is dominant. By Corollary 2.1.8.1 applied to the commutative diagram (4.12), the injection $G'^{r'} \rightarrow G^r$ of finite p -groups is an isomorphism.

In the case (2), by Proposition 1.3.5.4 (2) \Rightarrow (1), the E' -linear mapping $N_{T/Q} \otimes_{\mathcal{O}_T} E' \rightarrow \mathfrak{m}_{K'}^n N_{T'/Q'} \otimes_{\mathcal{O}_{T'}} E'$ is an injection. Hence the right vertical arrow of (4.12) induces a surjection $Q_{\bar{F}'}^{l[r']\circ} \rightarrow Q_{\bar{F}}^{[r]\circ} \times_{\bar{F}} \bar{F}'$. Hence by Corollary 2.1.8.3, the injection $G'^{r'} \rightarrow G^r$ is an isomorphism. The equivalence of the additivities also follows from Corollary 2.1.8.3 and Corollary 4.1.2.

In the case (3), since the diagram (4.11) is cartesian, the right vertical arrow of (4.12) induces an isomorphism $Q_{\bar{F}'}^{l[r']\circ} \rightarrow Q_{\bar{F}}^{[r]\circ} \times_{\bar{F}} \bar{F}'$. Hence the assertion follows from Corollary 2.1.8.3 and Corollary 4.1.2 as in the case (2). \square

Corollary 4.2.6. *Let the assumption be the same as in Proposition 4.2.5 except that we do not assume that the ramification of L is bounded by $r+$. Then, the canonical injection $G' \rightarrow G$ induces isomorphisms $\text{Gr}^{r'} G \rightarrow \text{Gr}^r G$ and $G'^{r'} \rightarrow G^r$.*

Proof. Let $M \subset L$ be the subextension corresponding to the subgroup $G^{r+} \subset G$ and $M' = MK' \subset L'$. Since the construction of the ramification groups is compatible with quotient [18, Corollary 1.4.3], the subquotient $\text{Gr}^r G = G^r/G^{r+}$ is identified with $H^r \subset \text{Gal}(M/K) = H = G/G^{r+}$. In the case (3), the morphism $\mathcal{O}_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \rightarrow \mathcal{O}_{M'}$ is an isomorphism by Lemma 4.2.4. Since the ramification of M over K is bounded by $r+$, Applying Proposition 4.2.5.1 to M and to $H' = \text{Gal}(MK'/K') = G'/G'^{r'+} \subset H = \text{Gal}(M/K) = G/G^{r+}$, we see that $\text{Gr}^{r'} G' \rightarrow \text{Gr}^r G$ is an isomorphism.

Hence for every rational number $s \geq r$ and $s' = es \geq r'$ in the cases (1) and (3) or $s' = es - m \geq r'$ in the case (2), the morphism $\text{Gr}^{s'} G' \rightarrow \text{Gr}^s G$ is an isomorphism. Thus $G'^{r'+} \rightarrow G^r$ is also an isomorphism. \square

4.3. Graded quotients.

Theorem 4.3.1. *Let K be a henselian discrete valuation field with residue field of characteristic $p > 0$ and let L be a finite Galois extension of K of Galois group G .*

1. *Let $r > 1$ be a rational number. Then, the graded quotient $\mathrm{Gr}^r G = G^r/G^{r+}$ is an abelian group and is an \mathbf{F}_p -vector space.*
2. *Let C be a field of characteristic different from p and let V be a representation of G on a C -vector space of finite dimension. Then, the total dimension $\dim_{\mathrm{tot}}(V)$ is an integer.*

Proof. In the classical case where the residue field is perfect, assertion 1 is proved in [22, Corollaire 1 of Proposition 7, Section 2, Chapitre IV] and assertion 2 is proved in [22, Théorème 1', Section 2, Chapitre VI]. Hence Theorem follows from the existence of tangentially dominant extension with perfect residue field Proposition 1.1.10 and the reduction step Corollary 4.2.6 (1). \square

For an abelian extension, we obtain the following.

Corollary 4.3.2. *Let the notation be as in Theorem 4.3.1 and assume that G is abelian. Let $r > 0$ be a rational number. If $G^r \supsetneq G^{r+}$, then r is an integer.*

Proof. It suffices to apply Theorem 4.3.1.2 to a character $\chi: G \rightarrow C^\times$ such that $\mathrm{Ker}\chi \supset G^{r+}$ and $\mathrm{Ker}\chi \not\supset G^r$. \square

Theorem 4.3.3. *Let K be a henselian discrete valuation field and let L be a finite Galois extension of K of Galois group G . Let $r > 1$ be a rational number and assume that the ramification of L over K is bounded by $r+$. Fix a morphism $i_0: L \rightarrow \bar{K}$ to a separable closure and let $X_{L/K, \bar{F}}^{(r)\circ} \subset X_{L/K, \bar{F}}^{(r)}$ and $\Theta_{L/K, \bar{F}}^{(r)\circ} \subset \Theta_{L/K, \bar{F}}^{(r)}$ (3.33) denote the connected components.*

1. *The G^r -torsor $X_{L/K, \bar{F}}^{(r)\circ}$ over $\Theta_{L/K, \bar{F}}^{(r)\circ}$ is additive.*
2. *The subgroup $G^r \subset G$ is an \mathbf{F}_p -vector space. The class $[X_{L/K, \bar{F}}^{(r)\circ}]$ defines an injection*

$$(4.13) \quad \mathrm{Hom}(G^r, \mathbf{F}_p) \rightarrow \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, \mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, \bar{F}))$$

of abelian groups, where $\mathcal{O}_L \rightarrow \bar{F}$ is induced by i_0 .

3. *Let $G_K = \mathrm{Gal}(\bar{K}/K) \rightarrow G = \mathrm{Gal}(L/K)$ be the surjection defined by $i_0: L \rightarrow \bar{K}$. Then, the injection (4.13) is compatible with $G_K \rightarrow G$ with respect to the conjugate action of G on the source and the natural action of G_K on the target.*

In the proof of Theorem 4.3.3, we do *not* use Theorem 4.3.1 or Proposition 1.1.10.

Proof. 1. First, we reduce the assertion to the case where $r > 1$ is an integer. By Proposition 4.2.5.2 (3) and Lemma 4.2.1, we may assume that K is complete by replacing K by the completion. Further by Proposition 4.2.5.2 (3) and Corollary 4.2.3, we may assume that there exists a discrete

valuation subring \mathcal{O}_{K_0} with perfect residue field $k = \mathcal{O}_{K_0}/\mathfrak{m}_{K_0}$ such that \mathcal{O}_K is essentially of finite type and generically separable over \mathcal{O}_{K_0} .

By Proposition 1.2.9 and Lemma 1.2.7.3, there exists a cartesian diagram

$$(4.14) \quad \begin{array}{ccc} Q_0 & \longleftarrow & T \\ \downarrow & & \downarrow \\ P_0 & \longleftarrow & S \end{array}$$

of schemes over $S_0 = \text{Spec } \mathcal{O}_{K_0}$ such that the vertical arrows are finite flat and the horizontal arrows are divisorial essentially immersions to smooth separated schemes. Let $m > 1$ be an integer such that $n = mr \geq 2$ is an integer and that the p -power part $q > 1$ of m satisfies $r \geq \frac{q}{q-1}$. Note that

$r > 1 = \lim_{q \rightarrow \infty} \frac{q}{q-1}$. We define a morphism $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ of discrete valuation rings as in Example 1.2.11. Hence by Proposition 4.2.5.2 (2), Proposition 1.3.5.4 and Example 1.3.6, by replacing K by K' , we may assume that $r = n > 1$ is an integer. By replacing K by an unramified extension, we may assume that the residue field E of L is a purely unramified extension of the residue field F of K .

We assume that $r = n \geq 2$ is an integer and that the residue field E of L is a purely inseparable extension of the residue field F of K . By Lemma 1.2.3.1 and Lemma 1.2.1.1, there exists a cartesian diagram

$$(4.15) \quad \begin{array}{ccc} Q & \longleftarrow & T \\ \downarrow & & \downarrow \\ P & \longleftarrow & S \end{array}$$

of schemes over S such that the vertical arrows are finite flat, the horizontal arrows are closed immersions to smooth schemes over S and that $T \rightarrow Q$ is minimal. Let F' be the residue field of the generic point ξ of the vector space $\Theta_{L/K, \bar{F}}^{(r)}$.

By Proposition 3.4.5, we obtain a commutative diagram

$$(4.16) \quad \begin{array}{ccc} X_{L/K, \bar{F}}^{(r)} \times_{\bar{F}} \bar{F}' & \longrightarrow & X_{L/K, \bar{F}}^{(r)} \times_{\bar{F}} \bar{F}' \\ \downarrow & & \downarrow \\ \Theta_{L/K, \bar{F}}^{(r)} \times_{\bar{F}} \bar{F}' & \xrightarrow{+\xi} & \Theta_{L/K, \bar{F}}^{(r)} \times_{\bar{F}} \bar{F}'. \end{array}$$

By the assumption that E is purely inseparable over F , the scheme $\Theta_{L/K, \bar{F}}^{(r)} = \Theta_{L/K, \bar{F}}^{(r)\circ}$ is connected. By replacing $f: T'_2 \rightarrow T_2$ in the construction of (4.16) in the proof of Proposition 3.4.5 by the composition of the automorphism of T_2 induced by an element of G , we may assume that the automorphism $X_{L/K, \bar{F}}^{(r)} \times_{\bar{F}} \bar{F}'$ (4.16) preserves the connected component $X_{L/K, \bar{F}}^{(r)\circ} \times_{\bar{F}} \bar{F}'$ defined by $i_0: L \rightarrow \bar{K}$.

Hence by Proposition 2.2.4, the scheme $X_{L/K, \bar{F}}^{(r)\circ}$ has a structure of group scheme over \bar{F} such that the finite étale morphism $X_{L/K, \bar{F}}^{(r)\circ} \rightarrow \Theta_{L/K, \bar{F}}^{(r)}$ of connected schemes is a morphism of group schemes. Consequently, by Proposition 2.1.6 (2) \Rightarrow (1), the G^r -torsor $X_{L/K, \bar{F}}^{(r)\circ} \rightarrow \Theta_{L/K, \bar{F}}^{(r)}$ is additive.

2. The assertion follows from 1. and Proposition 2.1.6 (1) \Rightarrow (3).

3. The actions of G_K and G on $\Theta_{L/K, \bar{F}}^{(r)}$ and on $X_{L/K, \bar{F}}^{(r)}$ defined by the action of G on L and the action of G_K on \bar{K} commute to each other. Let $\tilde{\tau} \in G_K$ and let $\tau \in G$ be its image. Then, the automorphisms $\tilde{\tau}^{-1} \circ \tau$ of $\Theta_{L/K, \bar{F}}^{(r)}$ and of $X_{L/K, \bar{F}}^{(r)}$ are compatible and preserve the connected components of $\Theta_{L/K, \bar{F}}^{(r)\circ}$ and of $X_{L/K, \bar{F}}^{(r)\circ}$. For $\sigma \in G$, we have a commutative diagram

$$\begin{array}{ccc} X_{L/K, \bar{F}}^{(r)\circ} & \xrightarrow{\sigma} & X_{L/K, \bar{F}}^{(r)\circ} \\ \tilde{\tau}^{-1} \circ \tau \downarrow & & \downarrow \tilde{\tau}^{-1} \circ \tau \\ X_{L/K, \bar{F}}^{(r)\circ} & \xrightarrow{\tau \sigma \tau^{-1}} & X_{L/K, \bar{F}}^{(r)\circ} \end{array}$$

Since the action of $\tilde{\tau}^{-1} \circ \tau$ on $\Theta_{L/K, \bar{F}}^{(r)\circ}$ is compatible with the action of $\bar{\tau}$ on $\text{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+}, \text{Tor}_1^{\mathcal{O}_L / \mathcal{O}_K}(\Omega_{\mathcal{O}_L / \mathcal{O}_K}^1, \bar{F}))$, the latter is compatible with the conjugate action of τ of G^r . \square

The injection (4.13) has the following functoriality. Let $K \rightarrow K'$ be an extension of henselian discrete valuation fields of ramification index e . Let $L \rightarrow L'$ be a morphism of finite Galois extensions of K and of K' and $G' = \text{Gal}(L'/K') \rightarrow G$ be the canonical morphism. Let $r' = er$ or $r' = er - m$ as in Lemma 3.3.5. Then, by the commutative diagram (3.35), the diagram

$$(4.17) \quad \begin{array}{ccc} \text{Hom}(G^r, \mathbf{F}_p) & \longrightarrow & \text{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+}, \text{Tor}_1^{\mathcal{O}_L / \mathcal{O}_K}(\Omega_{\mathcal{O}_L / \mathcal{O}_K}^1, E)) \\ \downarrow & & \downarrow \\ \text{Hom}(G^{r'}, \mathbf{F}_p) & \longrightarrow & \text{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}'}^{r'} / \mathfrak{m}_{\bar{K}'}^{r'+}, \text{Tor}_1^{\mathcal{O}_{L'} / \mathcal{O}_{K'}}(\Omega_{\mathcal{O}_{L'} / \mathcal{O}_{K'}}^1, E')) \end{array}$$

is commutative.

Corollary 4.3.4. *Let K be a henselian discrete valuation field and let L be a finite Galois extension of K of Galois group G . Let $r > 1$ be a rational number and fix a morphism $i_0: L \rightarrow \bar{K}$ to a separable closure.*

1. *The graded quotient $\text{Gr}^r G$ is an \mathbf{F}_p -vector space and the injection (4.13) defines an injection*

$$(4.18) \quad \text{Hom}(\text{Gr}^r G, \mathbf{F}_p) \rightarrow \text{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+}, \text{Tor}_1^{\mathcal{O}_L / \mathcal{O}_K}(\Omega_{\mathcal{O}_L / \mathcal{O}_K}^1, \bar{F})).$$

2. *Let $G_K = \text{Gal}(\bar{K}/K) \rightarrow G = \text{Gal}(L/K)$ be the surjection defined by $i_0: L \rightarrow \bar{K}$. Then, the injection (4.18) is compatible with $G_K \rightarrow G$ with respect to the conjugate action of G on the source and the natural action of G_K on the target.*

3. *The action of the wild inertia group $P = G^{1+}$ on $\mathrm{Gr}^r G$ is trivial. If $\mathrm{Gr}^r G \neq 1$, the prime-to- p part of the denominator of r divides the ramification index $e = e_{L/K}$.*

Proof. 1. Let $M \subset L$ be the intermediate extension corresponding to the subgroup $G^{r+} \subset G$ and identify $G' = \mathrm{Gal}(M/K) = G/G^{r+}$. Then since the filtrations are compatible with quotients by [18, Corollary 1.4.3], the composition of the injection (4.13) for M and for $G^{r+} = \mathrm{Gr}^r G$ with the injection $\mathrm{Tor}_1^{\mathcal{O}_M}(\Omega_{\mathcal{O}_M/\mathcal{O}_K}^1, \bar{F}) \rightarrow \mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, \bar{F})$ (1.38) defines an injection

$$\begin{aligned} \mathrm{Hom}(\mathrm{Gr}^r G, \mathbf{F}_p) &= \mathrm{Hom}(G^{r+}, \mathbf{F}_p) \rightarrow \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, \mathrm{Tor}_1^{\mathcal{O}_M}(\Omega_{\mathcal{O}_M/\mathcal{O}_K}^1, \bar{F})) \\ &\rightarrow \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, \mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, \bar{F})). \end{aligned}$$

2. This follows from Theorem 4.3.3.3.

3. Since the wild inertia subgroup $P_K \subset G_K$ acts trivially on the source of (4.18), the action of the wild inertia group P on $\mathrm{Gr}^r G$ is trivial by the injection (4.18) and 2.

Assume $\mathrm{Gr}^r G \neq 1$ and let m be the prime-to- p part of the denominator of r . Then, the action of the tame inertia group I_K/P_K on the image of $\mathrm{Gr}^r G$ induces an faithful action of the quotient μ_m . Hence by 2, μ_m is a quotient of the tame inertia I/P and m divides $e = e_{L/K}$. \square

By the functoriality (4.17), the injection (4.18) satisfies a similar functoriality. Namely, for an extension of henselian discrete valuation fields, we have a commutative diagram

$$(4.19) \quad \begin{array}{ccc} \mathrm{Hom}(\mathrm{Gr}^r G, \mathbf{F}_p) & \longrightarrow & \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, \mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, \bar{F})) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathrm{Gr}^{r'} G', \mathbf{F}_p) & \longrightarrow & \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}'}^{r'}/\mathfrak{m}_{\bar{K}'}^{r'+}, \mathrm{Tor}_1^{\mathcal{O}_{L'}}(\Omega_{\mathcal{O}_{L'}/\mathcal{O}_{K'}}^1, \bar{F})) \end{array}$$

for $r' = er$ or $r' = er - m > 1$ as in Lemma 3.3.5.

The composition of (4.18) with the injection $\mathrm{Tor}_1^{\mathcal{O}_L}(\Omega_{\mathcal{O}_L/\mathcal{O}_K}^1, \bar{F}) \rightarrow H_1(L_{\bar{F}/\mathcal{O}_K})$ (1.36) defines an injection

$$(4.20) \quad \mathrm{Hom}(\mathrm{Gr}^r G, \mathbf{F}_p) \rightarrow \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, H_1(L_{\bar{F}/\mathcal{O}_K})).$$

The morphism (4.20) satisfies a similar functoriality as (4.19). Taking the limit on L , we obtain an injection

$$(4.21) \quad \mathrm{Hom}(\mathrm{Gr}^r G_K, \mathbf{F}_p) \rightarrow \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, H_1(L_{\bar{F}/\mathcal{O}_K}))$$

for the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$.

If S is essentially of finite type and generically separable over $S_0 = \mathrm{Spec} \mathcal{O}_{K_0}$, the composition of (4.20) with the morphism $H_1(L_{E/S}) \rightarrow \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_K} E$ (1.12) defines a morphism

$$(4.22) \quad \mathrm{Hom}(\mathrm{Gr}^r G, \mathbf{F}_p) \rightarrow \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r/\mathfrak{m}_{\bar{K}}^{r+}, \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \bar{F}).$$

This is an injection if $e_{K/K_0} \neq 1$ by Proposition 1.1.5.3. For $r' = er$ or $r' = er - m > 1$ as in Lemma 3.3.5, we have a commutative diagram

$$(4.23) \quad \begin{array}{ccc} \mathrm{Hom}(\mathrm{Gr}^r G, \mathbf{F}_p) & \longrightarrow & \mathrm{Hom}_{\bar{F}}(\mathfrak{m}_{\bar{K}}^r / \mathfrak{m}_{\bar{K}}^{r+}, \Omega_{S/S_0}^1 \otimes_{\mathcal{O}_S} \bar{F}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathrm{Gr}^{r'} G', \mathbf{F}_p) & \longrightarrow & \mathrm{Hom}_{\bar{F}'}(\mathfrak{m}_{\bar{K}'}^{r'} / \mathfrak{m}_{\bar{K}'}^{r'+}, \Omega_{S'/S'_0}^1 \otimes_{\mathcal{O}_{S'}} \bar{F}'). \end{array}$$

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