Almost representations of algebras and quantization

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Abstract

We introduce the notion of almost representations of Lie algebras and quantum tori, and establish an Ulam-stability type phenomenon: every irreducible almost representation is close to a genuine irreducible representation. As an application, we prove that geometric quantizations of the two-dimensional sphere and the two-dimensional torus are conjugate in the semi-classical limit up to a small error.

Contents

[1]Partially supported by the European Research Council Starting grant 757585
[2]Partially supported by the Israel Science Foundation grant 1102/20

1 Introduction and main results

The goal of this article is twofold. The first objective is to establish an Ulam-stability type phenomenon for almost representations of algebras such as compact Lie algebras and quantum tori: we show that under certain assumptions, every irreducible almost representation of such an algebra is close to a genuine irreducible representation. In the case of Lie algebras, we present two versions of an Ulam-stability, associated to two different notions of an almost representation.

Remark 1.1. While a similar problem has been studied for representation of groups [13, 21, 10], to the best of our knowledge it has not been addressed in the framework of representations of algebras.

Our second goal is an application of the results on stability to geometric quantization.

1.1 Almost representations

Let us pass to a more detailed discussion of almost representations in the three cases we consider.

First case: For a finite-dimensional Hilbert space H, write $\|\cdot\|_{op}$ for the operator norm on the space $\mathfrak{su}(H)$ of skew-Hermitian operators acting on H. Recall that the Lie algebra $\mathfrak{su}(2)$ has real dimension 3, and admits a basis L_1, L_2, $L_3 \in \mathfrak{su}(2)$ satisfying the commutation relations

$$[L_j, L_{j+1}] = L_{j+2} \quad \text{for all} \quad j \in \mathbb{Z}/3\mathbb{Z}\,. \tag{1.1}$$

An *irreducible representation* is a linear map $\rho : \mathfrak{su}(2) \to \mathfrak{su}(H)$ preserving the commutation relations and such that the triple of skew-Hermitian operators $X_j := \rho(L_j)$, $j \in \mathbb{Z}/3\mathbb{Z}$, does not preserve any proper subspace of H. As well known in such a case, writing $n := \dim H$ for the complex dimension of H, we have

$$X_1^2 + X_2^2 + X_3^2 = -\frac{n^2-1}{4}\mathbb{1}\,. \tag{1.2}$$

Our first main result is as follows.

Theorem 1.2. *Fix $r > 0$, and for every $k \in \mathbb{N}$ and $c \in \mathbb{R}$, consider the following assumptions on a finite-dimensional Hilbert space H and a triple of operators $x_j \in \mathfrak{su}(H), j \in \mathbb{Z}/3\mathbb{Z}$:*

$$\left\| x_1^2 + x_2^2 + x_3^2 + \left(\tfrac{k^2}{4} + \tfrac{kc}{2}\right) \mathbb{1} \right\|_{op} \leq r\,; \tag{R1}$$

$$\left\| [x_j, x_{j+1}] - x_{j+2} \right\|_{op} \leq r/k \quad \textit{for all} \quad j \in \mathbb{Z}/3\mathbb{Z}\,; \tag{R2}$$

$$\dim H < 2(k+c)\,. \tag{R3}$$

Then the following holds:

1. *For any $c \notin \mathbb{Z}$, there exists $k_0 \in \mathbb{N}$ such that the system of assumptions $(R1)$ and $(R2)$ cannot be fullfilled for $k \geq k_0$.*

2. *For any $c \in \mathbb{Z}$, there exists $k_0 \in \mathbb{N}$ and $C_1, C_2 > 0$, such that for all $k \geq k_0$, for any finite dimensional H and triple of operators $x_j \in \mathfrak{su}(H), j \in \mathbb{Z}/3\mathbb{Z}$, satisfying $(R1), (R2)$ and $(R3)$, one has*

$$\dim H = k + c\,. \tag{1.3}$$

   Furthermore,

$$k/2 - C_1 \leq \|x_j\|_{op} \leq k/2 + C_1 \quad \textit{for all} \quad j \in \mathbb{Z}/3\mathbb{Z}\,, \tag{1.4}$$

   and there exists an irreducible representation $\rho : \mathfrak{su}(2) \to \mathfrak{su}(H)$ satisfying

$$\|x_j - \rho(L_j)\|_{op} \leq C_2 \quad \textit{for all} \quad j \in \mathbb{Z}/3\mathbb{Z}\,. \tag{1.5}$$

The proof of this theorem is given in Section 2. Let us point out that only the inequalities $(R1)$ and $(R2)$ are needed to establish (1.4). Note also that for genuine irreducible representations of $\mathfrak{su}(2)$, assumption $(R1)$ holds with $r = 1/4$ by (1.2), while $(R2)$ is valid for any $r > 0$ by (1.1).

Remark 1.3. Theorem 1.2 shows in particular that for $k \in \mathbb{N}$ big enough, any triple of operators x_1, x_2, $x_3 \in \mathfrak{su}(H)$ with $\dim H < 2(k+c)$ satisfying (R1) and (R2) acts irreducibly, i.e., has no common proper invariant subspace $V \subset H$. Conversely, the direct sum of two k-dimensional irreducible representations satisfy the assumptions (R1) and (R2) for $c = 0$, so that the assumption (R3) is optimal in order to get irreducible representations. For a related discussion on almost irreducibility, see the paragraph after Remark 3.3.

Conjecture 1.4. *The analogue of Theorem 1.2 holds for all real compact Lie algebras, with the appropriate relation in the left hand side of (R1) given by the Casimir element.*

Our proof of Theorem 1.2 uses the explicit description of representations of $\mathfrak{su}(2)$ for all $k \in \mathbb{N}$, and new ideas are needed in order to find a uniform proof for all compact Lie algebras in this setting.

SECOND CASE: The next result is a counterpart of Theorem 1.2 for *two-dimensional quantum tori*. Following [16, 27], recall that *the quantum torus* $\mathcal{A}_\theta$ is a C^*-algebra over $\mathbb{C}$ depending on a parameter $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. It is generated by two elements $W_1, W_2 \in \mathcal{A}_\theta$ with relations

$$W_1^* W_1 = W_2^* W_2 = \mathbb{1} \qquad \text{and} \qquad W_1 W_2 = e^{2\pi i\theta} W_2 W_1\,. \tag{1.6}$$

A $*$-*representation* $\rho : \mathcal{A}_\theta \to \mathrm{End}(H)$ on a Hilbert space H is then determined by the data of two unitary operators $X_1 := \rho(W_1)$, $X_2 := \rho(W_2) \in \mathrm{End}(H)$ satisfying $X_1 X_2 = e^{2\pi i\theta} X_2 X_1$. The C^*-algebra $\mathcal{A}_\theta$ admits an irreducible finite-dimensional $*$-representation whenever $e^{2\pi i\theta}$ is an n-th prime root of unity, and in that case we have $\dim H = n$.

Theorem 1.5. *Fix $r > 0$, and for any $c \in \mathbb{R}$ and $k \in \mathbb{N}$, consider the following assumptions on a finite-dimensional Hilbert space H and a pair of operators x_1, $x_2 \in \mathrm{End}(H)$:*

(R1) $$\left\| x_j x_j^* - \mathbb{1} \right\|_{op} \le r/k^3 \quad \textit{for all} \quad j = 1,\, 2\,;$$

(R2) $$\left\| \, x_1 x_2 - e^{2i\pi/(k+c)} x_2 x_1 \, \right\|_{op} \le r/k^3\,;$$

(R3) $$\dim H < 2(k + c)\;.$$

Then the following holds:

1. *For any $c \notin \mathbb{Z}$, there exists $k_0 \in \mathbb{N}$ such that the system of assumptions* $(R1)$, $(R2)$ *and* $(R3)$ *cannot be fullfilled for $k \ge k_0$.*

2. *For any $c \in \mathbb{Z}$, there exists $k_0 \in \mathbb{N}$ and $C > 0$ such that for all $k \ge k_0$, for any finite dimensional H and a pair of operators x_1, $x_2 \in \mathrm{End}(H)$, satisfying* $(R1)$, $(R2)$ *and* $(R3)$*, one has*

$$\dim H = k + c\,. \tag{1.7}$$

Furthermore, there exists a $*$*-representation* $\rho : \mathcal{A}_\theta \to End(H)$ *with* $\theta = 1/(k+c)$ *such that*

$$\|x_j - \rho(W_j)\|_{op} \leq C/k^{3/2} \quad \textit{for all} \quad j = 1,\, 2\,. \tag{1.8}$$

The proof follows the same strategy as the proof of Theorem 1.2, see Section 2.2 below.

Third case: In Section 3, we consider another notion of an irreducible almost representation $t : \mathfrak{g} \to \mathfrak{su}(H)$ of a compact Lie algebra $\mathfrak{g}$. It involves the Casimir element in the *adjoint representation*. As explained in Remark 3.3, this definition is weaker than the previous notion of an almost representation in the case of the Lie algebra $\mathfrak{su}(2)$, which is necessary for applications to geometric quantization of the two-sphere.

Take any orthonormal basis $e_1, \ldots, e_n$ in $\mathfrak{g}$ with respect to the Killing form. We define, in the context of almost representations, a counterpart of the Casimir element in the adjoint representation, called *almost-Casimir*, by

$$\begin{aligned} \Gamma : \mathfrak{su}(H) &\longrightarrow \mathfrak{su}(H) \\ \sigma &\longmapsto -\sum_{i=1}^{n} [[\sigma, t(e_i)], t(e_i)]\ . \end{aligned} \tag{1.9}$$

We define almost representations as linear maps $t : \mathfrak{g} \to \mathfrak{su}(H)$ which satisfy approximate commutation relations, and such that Γ is invertible. Theorem 3.2 below provides an upper bound for the distance between such a t and a genuine irreducible representation of $\mathfrak{g}$ in terms of the operator norm of the inverse of Γ. Here we adapt a Newton-type method as in [21].

1.2 Preliminaries on quantization

Geometric quantization is a mathematical recipe behind the quantum-classical correspondence, a fundamental physical principle stating that quantum mechanics contains classical mechanics in the limiting regime when the Planck constant $\hbar$ tends to zero. In Section 4, we apply Theorems 1.2 and 1.5 to show that all geometric quantizations of the two-dimensional sphere and the torus, satisfying the axioms of Definition 1.6 below, are conjugate to each other up to an error of order $\mathcal{O}(\hbar)$.

When the classical phase space is represented by a closed (i.e., compact without boundary) symplectic manifold (M, ω), geometric quantization is a linear correspondence $f \mapsto T_\hbar(f)$ between classical observables, i.e., real functions $f \in C^\infty(M)$ on the phase space M, and quantum observables, i.e., Hermitian operators $T_\hbar(f) \in \mathcal{L}(H_\hbar)$ on a complex Hilbert space $H_\hbar$. This correspondence is assumed to respect, in the leading order as the *Planck constant* $\hbar$ tends to 0, a number of basic operations.

Write $\{\cdot,\cdot\}$ for the *Poisson bracket* of (M, ω), defined on any f, $g \in C^\infty(M)$ by

$$\{f, g\} := -\omega(\mathrm{sgrad} f, \mathrm{sgrad} g)\,,$$

where for any $f \in C^\infty(M)$, the associated *Hamiltonian vector field* $\mathrm{sgrad} f$ over M is defined by

$$\omega(\cdot\,, \mathrm{sgrad} f) = df\,.$$

For any complex valued function $f \in C^\infty(M, \mathbb{C})$, we write $\|f\|_\infty := \max_{x\in M}\,|f|$ for its uniform norm.

Definition 1.6. Given a sequence $\{H_k\}_{k\in\mathbb{N}}$ of finite-dimensional complex Hilbert spaces, an associated *geometric quantization* is a collection of $\mathbb{R}$-linear maps $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ with $T_k(1) = \mathbb{1}$ for all $k \in \mathbb{N}$, satisfying the following axioms as $k \to +\infty$,

(P1) $$\|T_k(f)\|_{\mathrm{op}} = \|f\|_\infty + \mathcal{O}(1/k)\,;$$

(P2) $$[T_k(f), T_k(g)] = \tfrac{i}{k} T_k\left(\{f, g\}\right) + \mathcal{O}(1/k^2)\,;$$

(P3) $$T_k(f)T_k(g) = T_k\left(fg + \tfrac{1}{k} C_1(f, g) + \tfrac{1}{k^2} C_2(f, g)\right) + \mathcal{O}(1/k^3)\,.$$

In axiom (P3), we extend $\{T_k : C^\infty(S^2, \mathbb{C}) \to \mathrm{End}(H_k)\}_{k\in\mathbb{N}}$ by $\mathbb{C}$-linearity, and the maps C_1, $C_2 : C^\infty(S^2) \times C^\infty(S^2) \to C^\infty(S^2, \mathbb{C})$ are bi-differential operators. The remainders are taken with respect to the operator norm, uniformly in the C^N-norms of f, $g \in C^\infty(S^2)$ for some $N \in \mathbb{N}$.

In Definition 1.6, the integer $k \in \mathbb{N}$ represents a *quantum number*, and should be thought as inversely proportional to the Planck constant. Then the limit $k \to +\infty$ describes the so-called *semi-classical limit*, when the scale gets so large that we recover the laws of classical mechanics from those of quantum mechanics. In particular, the axiom (P2) is the celebrated *Dirac condition*, relating the Poisson bracket on classical observables to the commutator bracket on quantum observables.

Example 1.7. In the case M admits a complex structure $J \in \mathrm{End}(TM)$ compatible with ω and the De Rham cohomology class $[\omega]/2\pi$ is integral, the existence of geometric quantizations was established by Bordemann, Meinrenken and Schlichenmaier [5], using the theory of Boutet de Monvel and Guillemin [6]. Their construction is called *Berezin-Toeplitz quantization*, and goes as follows. Let L be a holomorphic Hermitian line bundle with Chern curvature equalt to $-i\omega$, and for any $k \in \mathbb{N}$, write L^k for the k-th tensor power of L. We define the Hilbert space H_k as the subspace of all global holomorphic sections of L^k inside the Hilbert space $L_2(M, L^k)$ of L_2-sections of L^k. With this language, the *Toeplitz operators* $T_k(f) \in \mathcal{L}(H_k)$ act by multiplication by $f \in C^\infty(M)$ composed with the orthogonal projection to H_k inside $L_2(M, L^k)$. In the case of the sphere S^2 and the torus $\mathbb{T}^2$, by an appropriate shift of the parameter $k \in \mathbb{N}$, this construction actually produces a discrete family of geometric quantizations depending on $m \in \mathbb{N}$ and satisfying $\dim H_k = k + m$.

While the construction given above is rather straightforward, verification of the axioms of Definition 1.6 is highly non-trivial. For comprehensive introductions to Berezin-Toeplitz quantization, see for instance [22, 23, 28].

The Berezin-Toeplitz quantizations associated to two distinct complex structures are essentially different, so that even for the simplest symplectic manifolds (M, ω), this construction produces a large variety of examples. As shown by Ma and Marinescu [24], Xu [30] and Charles [9], for such quantizations, the bi-differential operator $C_1(f, g)$ is proportional to the Hermitian product of the Hamiltonian vector fields of f and g, while the bi-differential $C_2(f, g)$ involves the Ricci curvature.

A different, albeit related, mathematical model of quantization is *deformation quantization* [3], which is an $\hbar$-linear associative algebra on the space $C^\infty(M)[[\hbar]]$ such that for all $f,\, g \in C^\infty(M)$,

$$f * g = fg + \hbar\, C_1(f, g) + \hbar^2\, C_2(f, g) + \cdots \,, \tag{1.10}$$

with $C_1(f, g) - C_1(g, f) = i\{f, g\}$. Here the Planck constant $\hbar$ plays the role of a formal deformation parameter, and the operation (1.10) is called a *star-product*. In Section 4.3, we consider geometric quantizations satisfying an extension of axiom (P3), given in Definition 4.6, to an asymptotic expansion up to $\mathcal{O}(1/k^m)$ for any $m \in \mathbb{N}$. This defines a star product via the formal relation $T_\hbar(f)T_\hbar(g) = T_\hbar(f * g)$ with $\hbar = 1/k$. In particular, the Berezin-Toeplitz quantizations described above satisfy this extension of axiom (P3), and thus induce a star-product [5, 29, 14] over (M,ω). While deformation quantizations of closed symplectic manifolds are completely classified up to *star equivalence* given by formula (4.68) below, the classification of geometric quantizations up to conjugation and an error of order $\mathcal{O}(1/k^m)$ with given $m \in \mathbb{N}$ is not yet understood. The study of this classification is the main motivation of this paper.

1.3 Applications to quantization

The second main result of this paper is as follows.

Theorem 1.8. *Asssume that* $(M,\omega) = S^2$ *or* $\mathbb{T}^2$ *endowed with the standard area form of volume* 2π. *Let* $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ *be a geometric quantization, and assume that*

$$\limsup_{k\to+\infty} \dim H_k/k < 2\,. \tag{1.11}$$

Then there exists an integer $m \in \mathbb{Z}$ *such that for all* $k \in \mathbb{N}$ *big enough, we have*

$$\dim H_k = k + m\,. \tag{1.12}$$

Furthermore, if $\{Q_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ *is another geometric quantization associated to the same sequence of Hilbert spaces, there exists a sequence of unitary operators* $\{U_k : H_k \to H_k\}_{k\in\mathbb{N}}$ *such that for any* $f \in C^\infty(M)$, *there exists* $C > 0$ *such that for any* $k \in \mathbb{N}$, *we have*

$$\|U_k^{-1} Q_k(f) U_k - T_k(f)\|_{op} \leq C/k\,. \tag{1.13}$$

From a physical point of view, property (1.13) is the statement that two different quantizations of the same classical phase space reproduce the same physics at the semi-classical limit, when $k \to +\infty$. Two geometric quantizations satisfying (1.13) are called *semi-classically equivalent.* Note on the other hand that for any $m \in \mathbb{N}$, a geometric quantization satisfying (1.12) can be realized through the construction of Example 1.7. We thus get the following corollary.

Corollary 1.9. *Under the dimension assumption* (1.11), *every geometric quantization of the sphere or the torus is semi-classically equivalent to a Berezin-Toeplitz quantization of Example 1.7.*

The semi-classical equivalence of the Berezin-Toeplitz quantization and the Kostant-Souriau quantization was first shown by Schlichenmaier in [29]. For Berezin-Toeplitz quantizations associated with a different compatible complex structure, Corollary 1.9 follows from the work of Charles [8]. These results were established for more general symplectic manifolds than the sphere and the torus, which leads to the following conjecture.

Conjecture 1.10. *Two geometric quantizations of a closed symplectic manifold (M, ω) associated with sequences of Hilbert spaces of the same dimension are semi-classically equivalent.*

In particular, it is not completely clear to what extent the dimension assumption (1.11) can be relaxed. An affirmative answer to Conjecture 1.4 should yield an affirmative answer to Conjecture 1.10 in the case of coadjoint orbits of general compact Lie groups, at least with the appropriate assumption on the dimension.

Remark 1.11. The dimension assumption (1.11) of Theorem 1.8 is natural from a physical point of view. In fact, equation (1.15) follows from an additional *trace axiom* for geometric quantizations, which is satisfied for Berezin-Toeplitz quantizations and which we discuss in Section 4.3. For geometric quantizations of closed symplectic manifolds (M, ω) with $\dim M = 2d$, this trace axiom implies the following estimate as $k \to +\infty$,

$$\dim H_k = \left(\frac{k}{2\pi}\right)^d \mathrm{Vol}(M, \omega) + \mathcal{O}(k^{d-1}) \ . \tag{1.14}$$

This reflects the physical principle that $\dim H_k$ approximately equals the maximal number of pair-wise disjoint quantum cells, i.e. cubes of volume $(2\pi\hbar)^d$, inside the classical phase space (M, ω). When $\dim M = 2$, formula (1.14) reads

$$\dim H_k = k + \mathcal{O}(1) \ , \tag{1.15}$$

so that the dimension assumption (1.11) holds in particular for geometric quantizations of $M = S^2$ or $\mathbb{T}^2$ satisfying the trace axiom.

In Theorem 4.4, we consider geometric quantization satisfying a *trace axiom*, given in Definition 4.3, and we express the asymptotics of the trace in terms of the bi-differential operator C_2 of axiom (P3). In Corollary 4.7, we show that this implies the equality of the usual trace with the *canonical* *trace* of the induced *star product* up to $\mathcal{O}(1/k)$, defined by formula (4.69) below. Finally, in Theorem 4.5, we show how Theorem 1.5 can be applied to get an extension of Theorem 1.8 for quantizations of the torus $\mathbb{T}^2$ which are invariant by translation, making a first step towards the classification of geometric quantizations up to order $\mathcal{O}(1/k^m)$ with $m > 1$.

2 Proofs for $\mathfrak{su}(2)$ and quantum torus

Let H be a Hilbert space of complex dimension $\dim H = n \in \mathbb{N}$, and recall that an operator $A \in \mathrm{End}(H)$ is called *normal* if it can be diagonalized in an orthonormal basis. The following Lemma on the existence of quasimodes will be a basic tool in this Section.

Lemma 2.1. *Let $A \in \mathrm{End}(H)$ be normal, and assume that v, $w \in H$, $v \neq 0$, and $\alpha \in \mathbb{C}$ satisfy*

$$Av = \alpha v + w\,. \tag{2.1}$$

Then there exists $\lambda \in \mathrm{Spec}(A)$ satisfying

$$|\lambda - \alpha| \leq \frac{\|w\|}{\|v\|}\,. \tag{2.2}$$

Furthermore, for any $\delta > 0$, let $V_\delta \subset H$ be the direct sum to the eigenspaces of eigenvalues $\eta \in \mathrm{Spec}(A)$ satisfying $|\eta - \alpha| < \delta$. Then there exists $e \in V_\delta$ with $\|e\| = 1$ such that

$$\Big\| v - \|v\| e \Big\| \leq 2\frac{\|w\|}{\delta}\,. \tag{2.3}$$

Proof. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis of H diagonalizing A, with complex eigenvalues $\{\lambda_j\}_{j=1}^n$. Consider v, $w \in H$ and $\alpha \in \mathbb{C}$ satisfying formula (2.1). Then we have

$$\|w\| = \|(A - \alpha)v\| \geq \min_{1 \leq j \leq n} |\lambda_j - \alpha|\, \|v\|\,, \tag{2.4}$$

which implies that there exists $1 \leq m \leq n$ such that λ_m satisfies formula (2.2).

Fix now $\delta > 0$. Then formula (2.1) implies

$$\begin{aligned} \|w\|^2 = \sum_{1\leq j\leq n} |\lambda_j - \alpha|^2 \, |\langle v, e_j\rangle|^2 \geq \sum_{|\lambda_j-\alpha|\geq\delta} |\lambda_j - \alpha|^2 \, |\langle v, e_j\rangle|^2 \\ \geq \delta^2 \left\| v - \sum_{|\lambda_m-\alpha|<\delta} \langle v, e_m\rangle \, e_m \right\|^2 , \end{aligned} \tag{2.5}$$

Write $\widetilde{e} := \sum_{|\lambda_m-\alpha|<\delta} \langle v, e_m\rangle \, e_m \in V_\delta$. Then this implies in particular that

$$\|w\| \geq \delta \|v - \widetilde{e}\| \geq \delta \left| \, \|v\| - \|\widetilde{e}\| \, \right| . \tag{2.6}$$

Taking $e := \widetilde{e}/\|\widetilde{e}\|$, we then get

$$\begin{aligned} \left\| v - \|v\| \, e \right\| &\leq \left\| v - \widetilde{e} \right\| + \left| \, \|v\| - \|\widetilde{e}\| \, \right| \\ &\leq 2 \frac{\|w\|}{\delta} \, . \end{aligned} \tag{2.7}$$

This proves the result. □

2.1 Case of the Lie algebra $\mathfrak{su}(2)$

A triple of skew-Hermitian operators X_1, X_2, $X_3 \in \mathfrak{su}(H)$ is said to *generate an irreducible representation of* $\mathfrak{su}(2)$ if they satisfy the commutation relations (1.1) and do not preserve any proper subspace of H. From the basic representation theory of $\mathfrak{su}(2)$, this is equivalent with the fact that

$$\begin{aligned} X_1^2 + X_2^2 + X_3^2 = -\left(\frac{n^2-1}{4}\right) \mathbb{1} \qquad \text{and} \\ [X_j, X_{j+1}] = X_{j+2} \qquad \text{for all} \quad j \in \mathbb{Z}/3\mathbb{Z} \, . \end{aligned} \tag{2.8}$$

Set the *ladder operators* to be

$$Y_\pm := \pm i X_1 + X_2 \in \mathrm{End}(H) \, . \tag{2.9}$$

Then there exists an orthonormal basis $\{e_j\}_{j=1}^n$ of H such that for all $m \in \mathbb{N}$ with $0 \leq m \leq n-1$, we have

$$\begin{aligned} X_3 \, e_{n-m} &= i \left(\frac{n-1}{2} - m\right) e_{n-m} \, , \\ Y_\pm \, e_{n-m} &= \mp \sqrt{\frac{n^2-1}{4} - \left(\frac{n-1}{2} - m\right)^2 \mp \left(\frac{n-1}{2} - m\right)} \, e_{n-m\pm1} \end{aligned} \tag{2.10}$$

Note that the $\mp$ sign in front of the square root in the the second line of (2.10) is a matter of convention, as one can pass to the opposite sign by a change of orthonormal basis $e_j \mapsto (-1)^j e_j$ for all $1 \leq j \leq n$.

Conversely, if we have operators X_3, Y_+, $Y_- \in \mathrm{End}(H)$ satisfying (2.10) in an orthonormal basis, then setting $X_1 := i(Y_- - Y_+)/2$ and $X_2 := (Y_- + Y_+)/2$, we get three operators X_1, X_2, $X_3 \in \mathfrak{su}(2)$ generating an irreducible representation of $\mathfrak{su}(2)$ on H.

Let us now compare some basic consequences of the axioms (R1) and (R2) of Theorem 1.2 with the basic theory of representations of $\mathfrak{su}(2)$ described at the beginning of the Section. For any $k \in \mathbb{N}$, introduce the ladder operators

$$y_\pm := \pm i x_1 + x_2 \in \mathrm{End}(H_k)\,, \tag{2.11}$$

which satisfy $y_\pm^* = -y_\mp$. Then axiom (R2) translates to

$$\big\| \pm i y_\pm - [x_3, y_\pm] \big\|_{op} = \mathcal{O}(1/k)\,. \tag{2.12}$$

On the other hand, one has

$$\begin{aligned} y_+ y_- &= x_1^2 + x_2^2 + i[x_1, x_2]\,, \\ y_- y_+ &= x_1^2 + x_2^2 - i[x_1, x_2]\,, \end{aligned} \tag{2.13}$$

so that axioms (R1) and (R2) imply

$$\begin{aligned} \left\| y_+ y_- + \frac{(k+c)^2}{4}\mathbb{1} + x_3^2 - i x_3 \right\|_{op} &= \mathcal{O}(1)\,, \\ \left\| y_- y_+ + \frac{(k+c)^2}{4}\mathbb{1} + x_3^2 + i x_3 \right\|_{op} &= \mathcal{O}(1)\,. \end{aligned} \tag{2.14}$$

Proof of Theorem 1.2. Let us fix $c \in \mathbb{R}$ and $r > 0$, and consider all triples of operators $x_1, x_2, x_3 \mathfrak{su}(H), j \in \mathbb{Z}/3\mathbb{Z}$ satisfying $(R1)$ and $(R2)$ for some finite-dimensional Hilbert space H. All the estimates in the proof are with respect to the Hilbert norm as $k \to +\infty$.

The proof will be divided into 3 steps. In Step 1, we use the ladder operators (2.11) to construct $k + c$ distinct eigenvectors of $x_3 \in \mathfrak{su}(H)$ with distinct eigenvalues, showing the first statement of Theorem 1.2 and the inequality (1.4). In Step 2, we show that, if these eigenvectors do not generate the whole Hilbert space H, we can repeat the construction of Step 1 to get another set of $k+c$ distinct eigenvectors of $x_3 \in \mathfrak{su}(H)$, thus establishing formula (1.3) on the dimension under the assumption (R3). In Step 3, we define a representation $\rho : \mathfrak{su}(2) \to \mathfrak{su}(H)$ via formulas (2.10) for the basis constructed in Step 1, and establish (1.5).

STEP 1: Let $\lambda_k \in \mathbb{R}$ be the highest eigenvalue of the Hermitian endomorphism $-ix_3 \in \mathrm{End}(H)$, and let $e_k \in H$ with $\|e_k\| = 1$ be such that

$$x_3 e_k = i\lambda_k e_k \,. \tag{2.15}$$

Using formula (2.12), we get the estimate

$$x_3(y_+ e_k) = i(\lambda_k + 1) y_+ e_k + \mathcal{O}(1/k) \,. \tag{2.16}$$

Applying Lemma 2.1 to $A = -ix_3$, $v = y_+ e_k$, $w = \mathcal{O}(1/k)$, and using the fact that $\lambda_k \in \mathbb{R}$ is the highest eigenvalue of $-ix_3$, we get the estimate $\|y_+ e_k\| = \mathcal{O}(1/k)$. Using now formula (2.14) and Cauchy-Schwartz inequality, this implies

$$\mathcal{O}(1/k^2) = \|y_+ e_k\|^2 = -\langle y_- y_+ e_k, e_k \rangle = \frac{(k+c)^2}{4} - \lambda_k^2 - \lambda_k + \mathcal{O}(1) \,, \tag{2.17}$$

which readily leads to the estimate

$$\lambda_k = \frac{k+c-1}{2} + \mathcal{O}(1/k) \,. \tag{2.18}$$

The strategy of the proof is based on a recursive estimate on the eigenvalues of $-ix_3$ using the lowering operators, which we describe now. We will use the elementary fact that for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$ and all $\lambda \in \mathbb{R}$, we have

$$-\frac{k+c-1}{2} + \epsilon < \lambda < \frac{k+c}{2} - \epsilon \quad \text{implies} \quad \frac{(k+c)^2}{4} - \lambda^2 + \lambda > \delta(k+c) \,. \tag{2.19}$$

Now if $f \in H$ with $\|f\| = 1$ is an eigenvector of $-ix_3$ with eigenvalue $\lambda \in \mathbb{R}$ satisfying (2.19), we can use formula (2.14) and Cauchy-Schwartz inequality to get

$$\begin{aligned} \|y_- f\|^2 = -\langle y_+ y_- f, f \rangle &= \frac{(k+c)^2}{4} - \lambda^2 + \lambda + \mathcal{O}(1) \\ &\geq \delta k + \mathcal{O}(1) \,. \end{aligned} \tag{2.20}$$

On the other hand, formula (2.12) implies

$$x_3(y_- f) = i(\lambda - 1)y_- f + \mathcal{O}(1/k)\,. \tag{2.21}$$

We can thus apply Lemma 2.1 with $A = -ix_3$, $v = y_- f$ and $w = \mathcal{O}(1/k)$ to get an eigenvector $\widetilde{f} \in H$ of $-ix_3$ with associated eigenvalue $\widetilde{\lambda} \in \mathbb{R}$ satisfying the recursive estimates

$$\widetilde{\lambda} = \lambda - 1 + \mathcal{O}(1/k^{3/2})\,. \tag{2.22}$$

We emphasize that the appearance of the exponent $3/2$ in the recursive estimate (2.22) is crucial for the rest of the proof.

Fix $\epsilon > 0$ small enough in (2.19), and recall the estimate (2.18) for the highest eigenvalue of $-ix_3$. Let us prove by induction that for all $m \in \mathbb{N}$ satisfying $0 \leq m < k+c$, there is a normalized eigenvector $e_{k-m} \in H$ of $-ix_3$ with associated eigenvalue $\lambda_{k-m} \in \mathbb{R}$ satisfying

$$\lambda_{k-m} = \lambda_k - m + m\,\mathcal{O}(1/k^{3/2})\,. \tag{2.23}$$

Note that this is trivially satisfied for $m = 0$. On the other hand, we see from (2.18) that for any $0 \leq m < k + c - 1$, the right hand side of (2.23) satisfies (2.19) as soon as $k \in \mathbb{N}$ is big enough. Thus if (2.23) is satisfied for some $0 \leq m < k + c - 1$, we can apply the recursive estimate (2.22) to $\lambda = \lambda_{k-m}$ to get

$$\lambda_{k-m-1} = \lambda_{k-m} - 1 + \mathcal{O}(1/k^{3/2})\,, \tag{2.24}$$

which implies (2.23) with m replaced by $m + 1$. This shows by induction that for all $m \in \mathbb{N}$ satisfying $0 \leq m < k + c$, we have

$$\lambda_{k-m} = \frac{k+c-1}{2} - m + \mathcal{O}(1/\sqrt{k})\,. \tag{2.25}$$

Let us now write $\lambda_- \in \mathbb{R}$ for the lowest eigenvalue of $-ix_3$. The argument leading to the estimate (2.18) using y_- instead of y_+ leads to the estimate

$$\lambda_- = -\frac{k+c-1}{2} + \mathcal{O}(1/k)\,. \tag{2.26}$$

Assume without loss of generality that $c \geq 0$. Suppose, on the contrary, that $c \notin \mathbb{Z}$. Then $m = \lfloor k + c \rfloor < m + c$. Applying (2.25), we get that λ_{k-m} is smaller than the lowest eigenvalue (2.26) for $k \in \mathbb{N}$ big enough. This contradiction shows that necessarily $c \in \mathbb{Z}$, which proves the first statement of Theorem 1.2. We also get that $\|x_3\|_{op} = k/2 + \mathcal{O}(1)$, and by symmetry of the assumptions (R1) and (R2) in $j \in \mathbb{Z}/3\mathbb{Z}$, this yields (1.4).

Step 2: Let us now assume that (R3) holds, in addition to (R1) and (R2). Our goal is to establish formula (1.3) on the dimension. By the first statement of Theorem 1.2 and through the shift $k \mapsto k + c$, we will assume without loss of generality that $c = 0$. Using the estimate (2.25), we get a set of eigenvalues of $-ix_3$ parametrized by $m \in \mathbb{N}$ with $0 \leq m \leq k-1$, which are pairwise distinct for $k \in \mathbb{N}$ big enough. This implies that $\dim H \geq k$.

To establish formula (1.3) with $c = 0$, let us consider $k \in \mathbb{N}$ big enough and assume on the contrary that $\dim H \geq k+1$. Let $E \subset H$ be the direct sum of 1-dimensional eigenspaces associated with each of the eigenvalues (2.25), so that $\dim E = k$ for $k \in \mathbb{N}$ big enough, and E is a proper subspace of H. In particular, there exists an eigenvalue $\mu \in \mathbb{R}$ of $-ix_3$ admitting an eigenvector $e_\mu \notin E$. Note that μ has to lie between the highest eigenvalue (2.18) and the lowest eigenvalue (2.26).

Furthermore, we claim that for any $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such for any $k \geq k_0$, there exists $0 \leq m_0 \leq k-1$ such that

$$|\mu - \lambda_{k-m_0}| < 2\epsilon\,. \tag{2.27}$$

Indeed, considering (2.19) and (2.25) with $c = 0$, we can use repeatedly the recursive estimate (2.22) as in Step 1 to produce eigenvalues

$$\mu_m := \mu - m + m\mathcal{O}(1/k^{3/2})\,, \tag{2.28}$$

for all $m \in \mathbb{N}$ such that $\mu_m > -\frac{k-1}{2} + \epsilon$. Assume, on the contrary, that (2.27) is not satisfied for some $0 \leq m_0 \leq k-1$. Then we can use the recursive estimate (2.22) one more time to produce an eigenvalue μ_- satisfying $\mu_- < -\frac{k-1}{2} - \epsilon$. Thus, μ_- is smaller than the lowest eigenvalue (2.26). This contradiction proves (2.27).

Now for any $m \in \mathbb{N}$ and $\theta > 0$, write

$$V_m(\theta) := \bigoplus_{|\lambda - \lambda_{k-m}| < \theta} E_\lambda\,, \tag{2.29}$$

where $E_\lambda := \{\, v \in H \mid -ix_3 v = \lambda v \,\}$ for all $\lambda \in \mathbb{R}$. By assumption, there exists an eigenvector $e_\mu \in H$ associated with μ which does not belong to the eigenspace associated with λ_{k-m_0} in $E \subset H$. Thus inequality (2.27) implies

$$\dim V_{m_0}(2\epsilon) \geq 2\,. \tag{2.30}$$

Note that for $k \in \mathbb{N}$ big enough, we have either $m_0 > 0$ or $m_0 < k-1$ (or both). Without loss of generality, let us assume that $m_0 < k-1$.

We claim that

$$\dim V_m(2\epsilon + m\mathcal{O}(1/k^{3/2})) \geq 2\,, \tag{2.31}$$

for every $m_0 \leq m \leq k-1$. The proof goes by induction in m. We have already seen that (2.31) holds true for $m = m_0$. Assume it is valid for some $m < k-1$. Then there exists an eigenvalue $\widetilde{\mu}_m \in \mathbb{R}$ of $-ix_3$ satisfying

$$|\lambda_{k-m} - \widetilde{\mu}_m| \leq 2\epsilon + m\mathcal{O}(1/k^{3/2})\ , \tag{2.32}$$

with associated eigenvector orthogonal to the subspace $E \subset H$. Applying the recursive estimate (2.22) and comparing with (2.24), we get an eigenvalue $\widetilde{\mu}_{m+1} \in \mathbb{R}$ of $-ix_3$, which satisfies

$$|\lambda_{k-m-1} - \widetilde{\mu}_{m+1}| < 2\epsilon + (m+1)\mathcal{O}(1/k^{3/2})\ . \tag{2.33}$$

Hence if $\widetilde{\mu}_{m+1} \neq \lambda_{k-m-1}$, the estimate (2.31) for $m+1$ holds automatically. We can thus assume without loss of generality that $\widetilde{\mu}_{m+1} = \lambda_{k-m-1}$. In this case, the recursive estimate (2.22) implies

$$|\lambda_{k-m} - \widetilde{\mu}_m| = \mathcal{O}(1/k^{3/2})\,. \tag{2.34}$$

Consider now two normalized orthogonal eigenvectors $f_1 \in E$ and $f_2 \notin E$ respectively associated with λ_{k-m} and $\widetilde{\mu}_m$. Applying the second part of Lemma 2.1 with $\delta = 2\epsilon$ to $y_- f_1$ and $y_- f_2$, we get vectors

$$\widetilde{f}_1,\ \widetilde{f}_2 \in V_{m+1}(2\epsilon + \mathcal{O}(1/k^{3/2})) \tag{2.35}$$

with $\|\widetilde{f}_1\| = \|\widetilde{f}_2\| = 1$ such that for $j = 1,\, 2$, we have

$$\Big\|\, y_- f_j - \ \|y_- f_j\| \widetilde{f}_j \,\Big\| = \mathcal{O}(1/k)\ . \tag{2.36}$$

On the other hand, using formula (2.14) and the fact that $\langle f_1, f_2 \rangle = 0$, we have

$$\langle y_- f_1, y_- f_2 \rangle = -\langle y_+ y_- f_1, f_2 \rangle = \mathcal{O}(1)\,. \tag{2.37}$$

Furthermore, formula (2.20) shows that for $j = 1,\, 2$, we have

$$\|y_- f_j\|^2 \geq \delta k + \mathcal{O}(1)\,. \tag{2.38}$$

This gives

$$\langle \widetilde{f}_1, \widetilde{f}_2 \rangle = \frac{1}{\|y_- f_1\|} \frac{1}{\|y_- f_2\|} \langle y_- f_1, y_- f_2 \rangle + \mathcal{O}(1/k^2) = \mathcal{O}(1/k^2)\,, \tag{2.39}$$

so that $\widetilde{f}_1,\ \widetilde{f}_2 \in V_{m+1}(2\epsilon + \mathcal{O}(1/k^{3/2}))$ are linearly independent for $k \in \mathbb{N}$ big enough. This finishes the proof of claim (2.31).

It follows from (2.31) that for all $m_0 \leq m \leq k-1$, we have

$$\dim V_m(2\epsilon + \mathcal{O}(1/\sqrt{k})) \geq 2\,. \tag{2.40}$$

On the other hand, if $m_0 > 0$, we can repeat the same process starting with $V_{m_0}(2\epsilon)$ using y_+ instead of y_- to get (2.40) for all $m \in \mathbb{N}$ with $0 \leq m \leq k-1$.

By definition (2.29), the subspaces $V_m(2\epsilon + \mathcal{O}(1/\sqrt{k}))$ are pairwise orthogonal for each $m \in \mathbb{N}$ as soon as $k \in \mathbb{N}$ is big enough, and we have

$$\dim \bigoplus_{0 \leq m \leq k-1} V_m(2\epsilon + \mathcal{O}(1/\sqrt{k}))) \geq 2k\,. \tag{2.41}$$

This contradicts the assumption (R3) for $c = 0$, and shows that $\dim H = k$, thus proving (1.3).

Step 3: We are now left with constructing a representation satisfying (1.5). Assuming without loss of generality that $c = 0$, the argument above shows in particular that all eigenvalues of $-ix_3$ are simple and satisfy formula (2.25) for all $m \in \mathbb{N}$ with $0 \leq m \leq k-1$. Using Lemma (2.1), we get a normalized eigenvector $e_{m-1} \in H$ of $-ix_3$ associated with λ_{m-1} satisfying

$$\begin{aligned} \langle y_- e_m, e_{m-1} \rangle &= \langle \|y_- e_m\| e_{m-1}, e_{m-1} \rangle + \mathcal{O}(1/k) \\ &= \|y_- e_m\| + \mathcal{O}(1/k)\,. \end{aligned} \tag{2.42}$$

Starting with any eigenvector e_k of $-ix_3$ associated with λ_k, we can then construct an orthonormal eigenbasis $\{e_j\}_{j=1}^k$ for x_3 associated to the sequence of eigenvalues $\{\lambda_j\}_{j=1}^k$ and satisfying formula (2.42) for all $1 \leq m \leq k$. Let us now note that for any $\lambda \in \mathbb{R}$, using in particular formula (2.19), we have that

$$-\frac{k-1}{2} + \epsilon < \lambda < \frac{k}{2} - \epsilon \quad \text{implies} \quad \left| \frac{d}{d\lambda} \left(\sqrt{\frac{k^2-1}{4} - \lambda^2 + \lambda} \right) \right| = \mathcal{O}(\sqrt{k})\,, \tag{2.43}$$

Via the first line of (2.20) and formula (2.25), this implies for all $1 \leq m \leq k$ that

$$\|y_- e_{k-m}\| = \sqrt{\frac{k^2-1}{4} - \left(\frac{k-1}{2} - m\right)^2 + \left(\frac{k-1}{2} - m\right)} + \mathcal{O}(1)\,. \tag{2.44}$$

On the other hand, for all $j \neq m-1$, using formula (2.12) and Cauchy-Schwartz inequality, we get

$$\begin{aligned} i\langle y_- e_m, e_j\rangle &= \langle [x_3, y_-] e_m, e_j\rangle + \mathcal{O}(1/k) \\ &= i(\lambda_m - \lambda_j)\,\langle y_- e_m, e_j\rangle + \mathcal{O}(1/k)\,. \end{aligned} \tag{2.45}$$

Now formula (2.18) implies that $|\lambda_j - \lambda_m - 1| \geq 1/2$ as soon as $j \neq m-1$ and $k \in \mathbb{N}$ big enough, so that (2.45) implies

$$\langle y_- e_m, e_j\rangle = \mathcal{O}(1/k) \quad \text{for} \quad j \neq m-1\,, \tag{2.46}$$

and we get analogous formulas for $y_+ = -y_-^*$ by definition (2.11).

In the orthonormal basis $\{e_j\}_{j=1}^k$ of H constructed above and following (2.10) and (2.8), let us now set

$$\begin{aligned} X_3\, e_{n-m} &:= i\left(\frac{k-1}{2} - m\right) e_{n-m}\,, \\ Y_\pm\, e_{n-m} &:= \mp\sqrt{\frac{k^2-1}{4} - \left(\frac{k-1}{2} - m\right)^2 \mp \left(\frac{k-1}{2} - m\right)}\, e_{n-m\pm 1}\,, \end{aligned} \tag{2.47}$$

for all $0 \leq m \leq k-1$. By the basic representation theory of $\mathfrak{su}(2)$ described at the beginning of the section and the definition (2.11) of $y_\pm$, to show Theorem 1.2, it suffices to show that $\|x_3 - X_3\|_{op} = \mathcal{O}(1)$ and $\|y_\pm - Y_\pm\|_{op} = \mathcal{O}(1)$. Now formula (2.23) implies immediately that

$$\|x_3 - X_3\|_{op} = \mathcal{O}(1/\sqrt{k})\,. \tag{2.48}$$

On the other hand, for all $1 \leq j,\, m \leq k$, formulas (2.42), (2.44), (2.46) and (2.47) yield

$$\begin{aligned} \langle (y_\pm - Y_\pm) e_j, e_m\rangle &= \mathcal{O}(1/k) \quad \text{for} \quad m \neq j \pm 1,\,, \\ \langle (y_\pm - Y_\pm) e_m, e_{m\pm 1}\rangle &= \mathcal{O}(1)\,. \end{aligned} \tag{2.49}$$

Decompose the matrix into $y_\pm - Y_\pm = A + B$, where all coefficients of A vanish except $A_{m,m\pm 1} = \mathcal{O}(1)$ for all $1 \leq m \leq k$ and where $B_{jm} = \mathcal{O}(1/k)$ for all $1 \leq j,\, m \leq k$. Then we readily get $\|A\|_{op} = \mathcal{O}(1)$, while by Cauchy-Schwartz we compute

$$\begin{aligned} \|B\|_{op}^2 &= \max_{\|v\|=1} \sum_{j=1}^{k} \left| \sum_{m=1}^{k} B_{jm} \langle e_m, v \rangle \right|^2 \\ &\leq k \max_{1 \leq j \leq k} \sum_{m=1}^{k} |B_{jm}|^2 \\ &\leq k^2 \mathcal{O}(1/k^2) = \mathcal{O}(1)\,. \end{aligned} \tag{2.50}$$

By the triangle inequality this gives

$$\|y_\pm - Y_\pm\|_{op} \leq \|A\|_{op} + \|B\|_{op} = \mathcal{O}(1)\,. \tag{2.51}$$

We get (1.5) for the representation defined by (2.47) as described in the beginning of the section. This concludes the proof of Theorem 1.2.

□

2.2 Case of the quantum torus

A pair of unitary operators $X_1,\, X_2 \in \mathrm{End}(H)$ *generates an irreducible representation of the quantum torus* $\mathcal{A}_{1/n}$ if it satisfies a commutation relation

$$X_1 X_2 = e^{2\pi i/n} X_2 X_1\,. \tag{2.52}$$

Diagonalizing X_1 in an orthonormal basis $\{e_m\}_{m=1}^n$ of H, we readily get that $X_1,\, X_2 \in \mathrm{End}(H)$ generate an irreducible representation of the quantum torus if and only if there exists $\theta_1,\, \theta_2 \in \mathbb{R}/n\mathbb{Z}$ such that

$$\begin{aligned} X_1 &:= \mathrm{diag}\left(e^{2\pi i \frac{\theta_1}{n}}, e^{2\pi i \frac{\theta_1+1}{n}}, \cdots, e^{2\pi i \frac{\theta_1+m}{n}}, \cdots, e^{2\pi i \frac{\theta_1+n-1}{n}} \right), \\ X_2 e_m &:= e^{2\pi i \frac{\theta_2}{n}} e_{m+1} \quad \text{for all} \quad m \in \mathbb{Z}/k\mathbb{Z}\,. \end{aligned} \tag{2.53}$$

Note that in this case X_1 and X_2 have no non-trivial common invariant subspace.

Proof of Theorem 1.5. All the estimates in the proof are with respect to the Hilbert norm as $k \to +\infty$, and only depend otherwise on $c \in \mathbb{R}$ and $r > 0$ in the statement of the Theorem.

Consider the polar decomposition $x_j = P_j U_j$, where $U_j \in \mathrm{End}(H)$ is unitary and $P_j \in \mathrm{End}(H)$ is positive Hermitian for each $j = 1, 2$. Then axiom (R1) is equivalent to $\|P_j^2 - \mathbb{1}\|_{op} = \mathcal{O}(1/k^3)$, which implies that $\|P_j - \mathbb{1}\|_{op} = \mathcal{O}(1/k^3)$. Using the submultiplicativity of the operator norm, we then see that the unitary parts $U_1, U_2 \in \mathrm{End}(H)$ also satisfy the axioms (R1) and (R2), and that $\|x_j - U_j\|_{op} = \mathcal{O}(1/k^3)$ for each $j = 1, 2$. We are thus reduced to the case of $x_1, x_2 \in \mathrm{End}(H)$ being unitary endomorphisms. In particular, they are normal and Lemma 2.1 applies.

The proof of Theorem 1.5 will be divided into 3 steps, following the structure of the proof of Theorem 1.2. In Step 1, we construct a set of eigenvectors for $x_1 \in \mathrm{End}(H)$ using $x_2 \in \mathrm{End}(H)$ as a ladder operator. Then the only notable difference with the proof of Theorem 1.2 is that in the case of Theorem 1.5, Statement 1 and formula (1.7) on the dimension depend on each other, and are established together in Step 2.

Step 1: Let $e \in H$ be an eigenvector of $x_1 \in \mathrm{End}(H)$, and write $\lambda_0 \in \mathbb{C}$ for the associated eigenvalue. Then axiom (R2) implies

$$x_1(x_2 e) = \lambda_0 e^{\frac{2\pi i}{k+c}} x_2 e + \mathcal{O}(1/k^3)\,. \tag{2.54}$$

As x_2 is unitary by assumption, we have $\|x_2 e\| = 1$, so that Lemma 2.1, with $A = x_1$, $v = x_2 e$ and $w = \mathcal{O}(1/k^3)$, implies the existence of a normalized eigenvector $e_1 \in H$ of x_1 with associated eigenvalue $\lambda_1 \in \mathbb{C}$ satisfying

$$\lambda_1 = \lambda_0 e^{\frac{2\pi i}{k+c}} + \mathcal{O}(1/k^3)\,. \tag{2.55}$$

We then obtain by induction eigenvalues $\lambda_m \in \mathbb{C}$ for all $0 \leq m < k + c$ satisfying

$$\begin{aligned} \lambda_m &= \lambda_0 e^{\frac{2\pi i m}{k+c}} + m\,\mathcal{O}(1/k^3) \\ &= \lambda_0 e^{\frac{2\pi i m}{k+c}} + \mathcal{O}(1/k^2)\,. \end{aligned} \tag{2.56}$$

As $|\lambda_m| = 1$ by unitarity, these eigenvalues are distinct for all $0 \leq m < k + c$ as soon as $k \in \mathbb{N}$ is big enough. In particular, we have $\dim H \geq k + c$ as soon as $k \in \mathbb{N}$ is big enough.

Step 2: Our goal now is to establish Statement 1 and formula (1.7) of Theorem 1.5. Note that if $c \notin \mathbb{Z}$, then $\lambda_0 \neq \lambda_0 e^{\frac{2\pi i \lfloor k+c \rfloor}{k+c}}$, and one can apply the construction of Step 1 once more to get distinct eigenvalues of the form

(2.56) for all $0 \leq m < k+c+1$, so that in particular $\dim H \geq k+c+1$ for $k \in \mathbb{N}$ big enough. Hence to show that $c \in \mathbb{Z}$ and $\dim H = k+c$, it suffices to show that $\dim H < k+c+1$.

Assume by contradiction that $\dim H \geq k+c+1$. Let $E \subset H$ be the direct sum of 1-dimensional eigenspaces associated with each of the eigenvalues (2.56) for all $0 \leq m < k+c$, so that $\dim E < k+c+1$, and E is a proper subspace of H. In particular, there exists an eigenvalue $\widetilde{\lambda}_0 \in \mathbb{C}$ of x_1 admitting an eigenvector $\widetilde{e}_0 \notin E$. Assume first that we have

$$\left|\widetilde{\lambda}_0 - \lambda_m\right| \geq \frac{\pi}{2k}\,, \tag{2.57}$$

for all $0 \leq m < k+c$. Then we can repeat the reasoning of Step 1 to construct by induction eigenvalues $\widetilde{\lambda}_m \in \mathbb{C}$ for all $0 \leq m < k+c$ satisfying

$$\widetilde{\lambda}_m = \widetilde{\lambda}_0 e^{\frac{2\pi i m}{k+c}} + \mathcal{O}(1/k^2)\,. \tag{2.58}$$

As $|\widetilde{\lambda}_m| = 1$ by unitarity, they are all distinct for all $0 \leq m < k+c$ as soon as $k \in \mathbb{N}$ is big enough, and (2.57) also implies that they are distinct from the set (2.56) for $k \in \mathbb{N}$ big enough. This implies in particular that $\dim H \geq 2(k+c)$, which contradicts the assumption of the Theorem.

Let us now consider the remaining case

$$\left|\widetilde{\lambda}_0 - \lambda_{m_0}\right| < \frac{\pi}{2k}\,, \tag{2.59}$$

for some $0 \leq m_0 < k+c$. For any $m \in \mathbb{N}$ and $\mu > 0$, write

$$V_m(\mu) := \bigoplus_{|\lambda - \lambda_m| < \mu} E_\lambda\,, \tag{2.60}$$

where $E_\lambda := \{\, v \in H \,|\, x_1 v = \lambda v \,\}$ for all $\lambda \in \mathbb{R}$. Now by assumption, there exists an eigenvector $\widetilde{e}_0 \in H$ of x_1 associated with $\widetilde{\lambda}_0$ which does not belong to a 1-dimensional eigenspace associated with λ_{m_0}. The assumption (2.59) translates to

$$\dim V_{m_0}\left(\frac{\pi}{2k}\right) \geq 2\,. \tag{2.61}$$

We claim that

$$\dim V_m\left(\frac{\pi}{2k} + m\mathcal{O}(1/k^3)\right) \geq 2\,, \tag{2.62}$$

for every $m_0 \leq m \leq k + c$. The proof goes by induction in m, the case $m = m_0$ being given by (2.61). Assume that it is valid for some $m < k - 1$. Then there exist an eigenvalue $\mu_m \in \mathbb{R}$ of x_1 satisfying

$$|\lambda_m - \mu_m| < \frac{\pi}{2k} + m\mathcal{O}(1/k^3)\,, \tag{2.63}$$

with associated eigenvector orthogonal to the subspace $E \subset H$. Then applying the recursive process of Step 1, we get an eigenvalue $\mu_{m+1} \in \mathbb{R}$ of x_1 satisfying

$$|\lambda_{m+1} - \mu_{m+1}| < \frac{\pi}{2k} + (m+1)\mathcal{O}(1/k^3)\,, \tag{2.64}$$

Hence if $\lambda_{m+1} \neq \mu_{m+1}$, the claim (2.62) for $m+1$ holds automatically. We can thus assume withou loss of generality that $\lambda_{m+1} = \mu_{m+1}$. In this case, the recursive process of Step 1 implies

$$|\lambda_m - \mu_m| = \mathcal{O}(1/k^3)\,. \tag{2.65}$$

Consider now two normalized orthogonal eigenvectors $f_1 \in E$ and $f_2 \notin E$ respectively associated with λ_m and μ_m. Applyting the second part of Lemma 2.1 with $w = \mathcal{O}(1/k^3)$ and $\delta = \pi/2k$ applied to $v = x_2 f_1$ and $x_2 f_2$ respectively, and using the unitarity of $x_2 \in \mathrm{End}(H)$, we get vectors

$$\widetilde{f}_1,\ \widetilde{f}_2 \in V_{m+1}\left(\frac{\pi}{2k} + \mathcal{O}(1/k^3)\right) \tag{2.66}$$

with $\|\widetilde{f}_1\| = \|\widetilde{f}_2\| = 1$ such that for all $j = 1, 2$, we have

$$x_2 f_j = \widetilde{f}_j + \mathcal{O}(1/k^2)\,. \tag{2.67}$$

This implies

$$\begin{aligned} \left\|\widetilde{f}_1 - \widetilde{f}_2\right\| &\geq \|f_1 - f_2\|\,(1 + \mathcal{O}(1/k^2)) = \sqrt{2}\,(1 + \mathcal{O}(1/k^2))\,, \\ \left\|\widetilde{f}_1 + \widetilde{f}_2\right\| &\leq \|f_1 + f_2\|\,(1 + \mathcal{O}(1/k^2)) = \sqrt{2}\,(1 + \mathcal{O}(1/k^2))\,, \end{aligned} \tag{2.68}$$

so that by (2.67) and the unitarity of x_2, the vectors $\widetilde{f}_1,\ \widetilde{f}_2 \in V_{m+1}\left(\frac{\pi}{2k} + \mathcal{O}(1/k^3)\right)$ are linearly independent for $k \in \mathbb{N}$ big enough. This finishes the proof of claim (2.62).

Now by definition (2.60), these subspaces are pairwise orthogonal for each $m \in \mathbb{N}$ as soon as $k \in \mathbb{N}$ is big enough, and we have

$$\dim \bigoplus_{0 \leq m < k+c-1} V_m\left(\frac{\pi}{2k} + m\mathcal{O}(1/k^3)\right) \geq 2(k+c)\,. \tag{2.69}$$

This contradicts the assumption $\dim H < 2(k+c)$, and proves that $c \in \mathbb{Z}$ and $\dim H = k + c$. Thus we established the first statement of the theorem and (1.7).

Step 3: We are now left with constructing a $*$-representation satisfying (1.8). Via the shift $k \mapsto k+c$, it suffices to consider the case $c = 0$. Given an eigenvector $e_m \in H$ of x_1 associated with $\lambda_m \in \mathbb{C}$ as in formula (2.56) and applying Lemma 2.1 with $A = x_1$, $v = x_2 e_m$ and $w = \mathcal{O}(1/k^3)$ as in formula (2.67), we can choose the eigenvector $e_{m+1} \in H$ of x_1 asociated with $\lambda_{m+1} \in \mathbb{C}$ such that

$$x_2 e_m = e_{m+1} + \mathcal{O}(1/k^2)\,. \tag{2.70}$$

Starting from an arbitrary eigenvector $e_0 \in H_m$ of x_1 associated with λ_0, we construct in this way an eigenbasis $\{e_m\}_{m=0}^{k-1}$ for u_k, and using Lemma 2.1 again, we get $\theta \in \mathbb{R}$ such that

$$x_2 e_{k-1} = e^{i\theta}\, e_0 + \mathcal{O}(1/k^2)\,. \tag{2.71}$$

Setting $f_m := e^{-i\theta m/k} e_m$ for all $m \in \mathbb{Z}/k\mathbb{Z}$ and working in the basis $\{f_m\}_{m=0}^{k-1}$ instead, set

$$\begin{aligned} X_1 &:= \operatorname{diag}\left(\lambda_0, \lambda_0 e^{2i\pi/k}, \cdots, \lambda_0 e^{2i\pi m/k}, \cdots, \lambda_0 e^{2i\pi(k-1)/k}\right)\,, \\ X_2 f_m &:= e^{i\theta/k} f_{m+1} \quad \text{for all} \quad m \in \mathbb{Z}/k\mathbb{Z}\,. \end{aligned} \tag{2.72}$$

By construction, the endomorphism $x_1 \in \operatorname{End}(H)$ is diagonal in the same basis than X_1 with eigenvalues given by formula (2.56), so that

$$\|X_1 - x_1\|_{op} = \mathcal{O}(1/k^2)\,. \tag{2.73}$$

Furthermore, Cauchy-Schwartz inequality together with formulas (2.70) and (2.71) imply

$$\|X_2 - x_2\|_{op} = \mathcal{O}(1/k^{3/2})\,. \tag{2.74}$$

Comparing with (2.53), we get (1.8), which completes the proof of the theorem.

□

3 Almost representations of compact Lie algebras

In this section, we propose an alternative notion of irreducibility of almost representations in the context of general compact Lie algebras, and present another version of the Ulam-type statement: irreducible almost-representations can be approximated by a genuine representation.

Let $(\mathfrak{g}, \{\cdot,\cdot\})$ be a real *compact* n-dimensional Lie algebra. This means that it is semi-simple and that its Killing form $\langle\cdot,\cdot\rangle$ is negative definite. Consider an orthonormal basis $\{e_j\}_{j=1}^n$ of $\mathfrak{g}$ such that for all $1 \leq j,\ k \leq n$, we have

$$\langle e_j, e_k\rangle = -\delta_{jk}\ . \tag{3.1}$$

Let H be a complex Hilbert space of finite dimension. Recall that $\|\cdot\|_{op}$ denotes the operator norm on the space $\mathfrak{su}(H)$ of skew-Hermitian operators. For an operator $A : \mathfrak{su}(H) \to \mathfrak{su}(H)$ we write $|||A|||$ for its operator norm with respect to the operator norm on $\mathfrak{su}(H)$.

Definition 3.1. A linear map $t : \mathfrak{g} \to \mathfrak{su}(H)$ is called a (μ, K, ϵ)-*almost representation* of $(\mathfrak{g}, \{\cdot,\cdot\})$ if the following assumptions hold:

- For all $1 \leq j,\ k \leq n$, the *defect*

$$\alpha_{jk} := t(\{e_j, e_k\}) - [t(e_j), t(e_k)] \tag{3.2}$$

  satisfies $\epsilon := \max_{j,k} \|\alpha_{jk}\|_{op}$;

- $K := \max_j \|t(e_j)\|_{op}$;

- The *almost-Casimir operator* Γ defined by (1.9) is invertible with $\mu := |||\Gamma^{-1}|||$.

Theorem 3.2. *Let $(\mathfrak{g}, \{\cdot,\cdot\})$ be a real semi-simple compact finite dimensional Lie algebra. Then for any $c > 0$, there exists a constant $\gamma > 0$ with the following property. Given any (μ, K, ϵ)-almost representation $t : \mathfrak{g} \to \mathfrak{su}(H)$ with $\epsilon \leq \gamma \min(\mu^{-2}K^{-2}, \mu^{-1}, 1)$, there exists a representation $\rho : \mathfrak{g} \to \mathfrak{su}(H)$ such that for all $1 \leq j \leq n$,*

$$\|t(e_j) - \rho(e_j)\|_{\mathrm{op}} \leq c\,\mu\,K\epsilon\ . \tag{3.3}$$

Remark 3.3. Although more general, this result has a number of drawbacks as compared to Theorem 1.2, in the case $\mathfrak{g} = \mathfrak{su}(2)$. First, it is unclear to us how to estimate μ in the case of geometric quantizations of the sphere. Second, even if we have an *ansatz* $\mu \sim 1$ and $\|x_j\| \sim k \sim \dim H$, as it should be for an irreducible k-dimensional representation, the existence of a nearby genuine representation is guaranteed only when the defect $\epsilon \lesssim k^{-2}$, as opposed to a less restrictive assumption $\epsilon \lesssim k^{-1}$ provided by Theorem 1.2.

Discussion on almost irreducibility: For representations, the invertibility of the adjoint Casimir Γ is equivalent to irreducibility. In fact, note that the definition of almost-Casimir given in (1.9) extends to any collection $X = \{x_1, \ldots, x_n\}$ of operators in $\mathfrak{su}(H)$ by the formula

$$\Gamma\sigma := -\sum_{i=1}^{n} [[\sigma, x_i], x_i] \ .$$

Furthermore,

$$\mathrm{tr}(\Gamma(\sigma)\,\sigma) = \sum_{i=1}^{n} \mathrm{tr}\left([\sigma, x_i]^2\right) \ , \tag{3.4}$$

and hence $\Gamma\sigma = 0$ if and only if σ commutes with all the operators from X. In particular, Γ is invertible if and only if the operators from X possess a common proper invariant subspace. With this in mind, we are going to compare $\mu(X) := |||\Gamma^{-1}|||$ with another quantity of geometric flavor which can be interpreted as a magnitude of irreducibility. Put

$$d(X) := \min_{\Pi} \max_{j} \|(\mathbb{1} - \Pi)\, x_j\, \Pi\|_{op} \ ,$$

where Π runs over all orthogonal projectors to proper subspaces $V \subset H$, and $j \in \{1, 2, 3\}$. Intuitively speaking, smallness of d yields that the corresponding subspace V is almost invariant.

To this end, denote by $\mathcal{X}$ the space of all collections X whose almost-Casimir is invertible. We say that two positive functions on $\mathcal{X}$ are *equivalent* if their ratio is bounded away from 0 and $+\infty$ by two constants which depend on $\dim H$.

Proposition 3.4. *The functions $\mu^{-1/2}$ and d are equivalent.*

Sketch of the proof: Denote by $\|A\|_2 := \sqrt{\mathrm{tr}(A^*A)}$ the Hilbert-Schmidt norm of an operator, and by $\lambda_1(X)$ the first eigenvalue of $-\Gamma$. The standard inequalities between the Hilbert-Schmidt norm and the operator norm imply that μ^{-1} is equivalent to λ_1. The claim follows from the inequalities

$$d(X)^2 \le C_1(k)\lambda_1(X) \;, \tag{3.5}$$

and

$$\lambda_1(X) \le C_2(k)d(X)^2 \;. \tag{3.6}$$

In order to prove inequality (3.5), take an eigenvector A of Γ with $\|A\|_2 = 1$ corresponding to the first eigenvalue. Since $\mathrm{tr}A = 0$, the spectrum of A can be written as the union of two clusters lying at distance at least $\sim k^{-2}$ apart. Let Π be the spectral projection corresponding to one of them. Since by (3.4) A almost commutes with x_j up to ϵ, one readily deduces from Lemma 2.1 on quasimodes that the image of Π is almost invariant under x_j. This yields (3.5).

Inequality (3.6) follows from the identity

$$-(\Gamma(\Pi), \Pi) = 2\sum_{i=1}^{n} \big\|[x_i, \Pi]\big\|_2^2 \;, \tag{3.7}$$

which holds true for every orthogonal projector Π.

The details of the argument are left to the reader. □

It would be interesting to find sharp bounds on the ratio of $\mu^{-1/2}$ and d in terms of $\dim H$. At the moment, we cannot compute them even for genuine irreducible representations.

Proof of Theorem 3.2: To simplify the notations, we will often write $x_j := t(e_j)$ for all $1 \le j \le n$. All the estimates in the proof are with respect to the operator norm of $\mathfrak{su}(H)$ and only depend on $(\mathfrak{g}, \{\cdot,\cdot\})$.

For a linear map $a : \mathfrak{g} \to \mathfrak{su}(H)$, define an *approximate* Elienberg-Chevalley coboundary $d_t a : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{su}(H)$ by

$$d_t a(g, h) := [t(g), a(h)] - t(h), a(g)] - a(\{g, h\}) \;.$$

The proof follows the Newton-type iterative process due to Kazhdan [21] adapted to the context of Lie algebras. At the first step we try to find a linear map $a : \mathfrak{g} \to \mathfrak{su}(H)$ so that

$$\bar{t}(g) := t(g) + a(g) \tag{3.8}$$

is a genuine representation. This yields equation

$$\alpha(g,h) - d_t a(g,h) - [a(g), a(h)] = 0\ . \tag{3.9}$$

Ignore the third, quadratic in a term, and solve the linearized equation $d_t a = \alpha$. As we will see, the almost representation $\bar{t} := t + a$ is closer to a genuine representation. Repeating the process, we get in the limit the desired genuine representation approximating the original almost representation t.

To make this precise, we have to solve the linearized homological equation $d_t a = \alpha$. This is done by using an effective approximate version of Whitehead's Lemma (see p.88–89 of [18]).

Consider the anti-symmetric 2-form $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{su}(H)$ defined for any g, $h \in \mathfrak{g}$ by

$$\alpha(g,h) := t(\{g,h\}) - [t(g), t(h)] \tag{3.10}$$

and the 1-form $a : \mathfrak{g} \to \mathfrak{su}(H)$ defined for any $g \in \mathfrak{g}$ by

$$a(g) := -\sum_{i=1}^{n} \Gamma^{-1}[\alpha(g, e_i), x_i]\,. \tag{3.11}$$

Lemma 3.5. *For all* $j, k = 1, \dots n$

$$\alpha(e_j, e_k) = d_t a(e_j, e_k) + O(\mu^2 K^2 \epsilon^2)\,. \tag{3.12}$$

The lemma is proved at the end of this section.

Let us now consider the linear map $\bar{t} : \mathfrak{g} \to \mathfrak{su}(H)$ defined for all $g \in \mathfrak{g}$ by

$$\bar{t}(g) := t(g) + a(g)\,, \tag{3.13}$$

and set $\bar{x}_j := \bar{t}(e_j)$ for all $1 \leq j \leq n$. Then for all $1 \leq j \leq n$, by formula (3.11) for $a(e_j)$ we have

$$\bar{x}_j \leq K(1 + O(\mu\,\epsilon))\,. \tag{3.14}$$

On the other hand, considering for all $1 \leq j$, $k \leq n$ the defect

$$\bar{\alpha}_{jk} := \bar{t}(\{e_j, e_k\}) - [\bar{t}(e_j), \bar{t}(e_k)]\,, \tag{3.15}$$

we see from (3.9) that

$$\bar{\alpha}_{jk} = \alpha_{jk} - d_t a(e_j, e_k) - [a(e_j), a(e_k)] = O(\mu^2 K^2 \epsilon^2)\,. \tag{3.16}$$

Finally, consider the almost-Casimir operator $\overline{\Gamma} : \mathfrak{su}(H) \to \mathfrak{su}(H)$ defined as in (1.9) with x_k replaced by $\overline{t}(e_k)$ for all $1 \leq k \leq n$. Then we get

$$\overline{\Gamma} = \Gamma + \epsilon(1+\mu\epsilon)\, O(\mu K^2) = \Gamma\Big(\mathbb{1} + \epsilon(1+\mu\epsilon)\, O(\mu^2 K^2)\Big)\,.$$

This implies that for any $\delta > 0$, there exists a constant $\gamma > 0$ such that if $\epsilon(1+\mu\epsilon) \leq \gamma/\mu^2 K^2$, then $\overline{\Gamma}$ is invertible and for all $\sigma \in \mathfrak{su}(H)$, its inverse satisfies

$$\left\|\overline{\Gamma}^{-1}(\sigma)\right\|_{\mathrm{op}} \leq (1+\delta)\mu\, \|\sigma\|_{op}\,.$$

This, together with the estimates (3.14) and (3.16), shows that for any $\delta > 0$, there exists $\gamma > 0$ such that under the hypothesis $\epsilon \leq \gamma \min(\mu^{-2}K^{-2}, \mu^{-1}, 1)$, the linear map $\overline{t} : \mathfrak{g} \to \mathfrak{su}(H)$ is an $(\overline{\mu}, \overline{K}, \overline{\epsilon})$-almost representation with

$$\overline{\mu} \leq \mu\,(1+\delta)\,, \quad \overline{K} \leq K\,(1+\delta) \quad \text{and} \quad \overline{\epsilon} \leq \epsilon\,\delta\,.$$

Taking $\delta > 0$ such that $\delta < (1+\delta)^{-4}$, we get that $\overline{\epsilon} \leq \gamma \min(\overline{\mu}^{-2}\overline{K}^{-2}, \overline{\mu}^{-1}, 1)$, and we can reiterate the construction above with the $(\overline{\mu}, \overline{K}, \overline{\epsilon})$-almost representation $\overline{t} : \mathfrak{g} \to \mathfrak{su}(H)$ instead of $t : \mathfrak{g} \to \mathfrak{su}(H)$. At the N-th iteration, we get a (μ_N, K_N, ϵ_N)-almost representation $t_N : \mathfrak{g} \to \mathfrak{su}(H)$ with

$$\mu_N \leq \mu\,(1+\delta)^N\,, \quad K_N \leq K\,(1+\delta)^N \quad \text{and} \quad \epsilon_N \leq \epsilon\,\delta^N\,.$$

Writing $a_N : \mathfrak{g} \to \mathfrak{su}(H)$ for the the 1-form defined as in (3.11) for $t_N : \mathfrak{g} \to \mathfrak{su}(H)$, for all $1 \leq j \leq n$ we get

$$\begin{aligned} t_N(e_j) &= t_{N-1}(e_j) + a_N(e_j) = t(e_j) + \sum_{k=1}^{N} a_k(e_j) \\ &= t(e_j) + \sum_{k=1}^{N} \big((1+\delta)^2\delta\big)^k O(\mu\, K\epsilon)\,, \end{aligned} \tag{3.17}$$

and the sum of the last line converges as $N \to +\infty$ for $\delta > 0$ small enough. As $\epsilon_N \to 0$, the limit map $\rho : \mathfrak{g} \to \mathfrak{su}(H)$ is a genuine representation, satisfying the inequality (3.3) by (3.17). □

Proof of Lemma 3.5: First note that by definition, for any $1 \leq i,\, j,\, k \leq n$, we have

$$\begin{aligned} 0 = [\alpha_{jk}, x_i] + \alpha(\{e_j, e_k\}, e_i) + [\alpha_{ki}, x_j] \\ + \alpha(\{e_k, e_i\}, e_j) + [\alpha_{ij}, x_k] + \alpha(\{e_i, e_j\}, e_k)\,. \end{aligned}$$

Taking the bracket of this identity with x_i and following the computations of [18, p.90], this implies that for all $1 \leq j,\, k \leq n$, we have

$$\Gamma\alpha_{jk} = \sum_{i=1}^{n} \Big([\alpha(\{e_j, e_k\}, e_i), x_i] + [[\alpha_{ki}, x_i], x_j] - [[\alpha_{ji}, x_i], x_k] \Big) - A_{jk} \,, \quad (3.18)$$

with

$$A_{jk} := -\sum_{i=1}^{n} \Big([\alpha_{ki}, [x_j, x_i]] - [\alpha(e_k, \{e_i, e_j\}), x_i] - [\alpha_{ji}, [x_k, x_i]] + [\alpha(e_j, \{e_i, e_k\}), x_i] \Big) \,. \quad (3.19)$$

Applying $\Gamma^{-1} : \mathfrak{su}(H) \to \mathfrak{su}(H)$ on both sides of the equality (3.18) and recalling the definition (3.11) of $a : \mathfrak{g} \to \mathfrak{su}(H)$, we get

$$\alpha_{jk} = [x_j, a(e_k)] - [x_k, a(e_j)] - a(\{e_i, e_j\}) - B_{jk} - \Gamma^{-1} A_{jk} \,, \quad (3.20)$$

with

$$B_{jk} := -\sum_{i=1}^{n} \Big(\Gamma^{-1}[[\alpha_{ki}, x_i], x_j] - [\Gamma^{-1}[\alpha_{ki}, x_i], x_j] - \Gamma^{-1}[[\alpha_{ji}, x_i], x_k] + [\Gamma^{-1}[\alpha_{ji}, x_i], x_k] \Big) \,. \quad (3.21)$$

Let us now estimate the terms (3.19) and (3.21). First note that as the Killing form $\langle \cdot, \cdot \rangle$ is Ad-invariant and by the explicit formula (3.1), we have

$$\begin{aligned} -\sum_{i=1}^{n} [\alpha(e_k, \{e_i, e_j\}), x_i] &= \sum_{i=1}^{n} \Big(\sum_{l=1}^{n} \langle \{e_i, e_j\}, e_l \rangle [\alpha_{kl}, x_i] \Big) \\ &= \sum_{l=1}^{n} \Big[\alpha_{kl}, \sum_{i=1}^{n} \langle \{e_j, e_l\}, e_i \rangle x_i \Big] = -\sum_{l=1}^{n} [\alpha_{kl}, t(\{e_j, e_l\})] \quad (3.22) \\ = -\sum_{l=1}^{n} [\alpha_{kl}, [x_j, x_l]] + \sum_{l=1}^{n} [\alpha_{kl}, \alpha_{jl}] &= -\sum_{l=1}^{n} [\alpha_{kl}, [x_j, x_l]] + O(\epsilon^2) \,. \end{aligned}$$

Comparing with formula (3.19) for A_{jk}, this implies that

$$\Gamma^{-1} A_{jk} = O(\mu\, \epsilon^2) \,. \quad (3.23)$$

On the other hand, following [18, p. 78] for any $g \in \mathfrak{g}$ and $1 \leq j \leq n$, using the Killing form in the same way than in (3.22) we get

$$\Gamma[g, x_j] = -\sum_{i=1}^{n}[[[g, x_j], x_i], x_i] = [\Gamma g, x_j] - \sum_{i=1}^{n}[[g, [x_j, x_i]], x_i] - \sum_{i=1}^{n}[[g, x_i], [x_j, x_i]]$$
$$= [\Gamma g, x_j] - C_j(g) - \sum_{i=1}^{n}[[g, t(\{e_j, e_i\})], x_i] - \sum_{i=1}^{n}[[g, x_i], t(\{e_j, e_i\})] = [\Gamma g, x_j] - C_j(g)\,,$$

with

$$C_j(g) := -\sum_{i=1}^{n}[[g, \alpha_{ji}], x_i] - \sum_{i=1}^{n}[[g, x_i], \alpha_{ji}]\,.$$

In particular, for any $1 \leq i,\, j,\, k,\, l \leq n$, we have

$$\begin{aligned} \Gamma^{-1}[[\alpha_{ij}, x_k], x_l] - [\Gamma^{-1}[\alpha_{ij}, x_k], x_l] \\ = \Gamma^{-1}\left([[\alpha_{ij}, x_k], x_l] - \Gamma[\Gamma^{-1}[\alpha_{ij}, x_k], x_l]\right) \\ = \Gamma^{-1}C_l(\Gamma^{-1}[\alpha_{ij}, x_k]) = O(\mu^2 K^2 \epsilon^2)\,. \end{aligned}$$

Comparing with formula (3.21) for B_{jk}, we thus get

$$B_{jk} = O(\mu^2 K^2 \epsilon^2)\,. \tag{3.24}$$

Then via the estimates (3.23) and (3.24), the identity (3.20) becomes

$$\alpha_{jk} = [x_j, a(e_k)] - [x_k, a(e_j)] - a(\{e_k, e_j\}) + O(\mu^2 K^2 \epsilon^2)\,.$$

This completes the proof of the lemma. □

4 Equivalence of quantizations

The basic strategy of the proofs of Theorem 1.8 is to show that geometric quantizations of the sphere or the torus induce almost representations of $\mathfrak{su}(2)$ and the quantum torus respectively, when restricted to a specific set of basic functions, and then use our Theorems 1.2 and 1.5. Let us first start with some generalities on geometric quantizations of a closed symplecic manifold (M, ω).

4.1 General setting

Let (M, ω) be a closed symplectic manifold. A bi-differential operator $C : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is called a *Hochschild cocycle* if for all $f_1,\ f_2,\ f_3 \in C^\infty(M)$, we have

$$\begin{aligned}\partial_H C(f_1, f_2.f_3) \\ := f_1\, C(f_2, f_3) - C(f_1 f_2, f_3) + C(f_1, f_2 f_3) - C(f_1, f_2)\, f_3 \\ = 0\,. \quad (4.1)\end{aligned}$$

The operator ∂_H is called the *Hochschild differential*. We will write

$$C_-(f,g) := \frac{C(f,g) - C(g,f)}{2} \quad \text{and} \quad C_+(f,g) := \frac{C(f,g) + C(g,f)}{2}\,.$$

for the anti-symmetric and symmetric part of C.

Assume now that $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$, satisfy the axioms of Definition 1.6. The associativity of composition of operators implies that the bi-differential C_1 appearing in axiom (P3) is a Hochschild cocycle, and that for any $f_1,\ f_2,\ f_3 \in C^\infty(M)$, we have

$$\partial_H C_2(f_1, f_2, f_3) = C_1(C_1(f_1, f_2), f_3) - C_1(f_1, C_1(f_2, f_3))\,. \tag{4.2}$$

Furthemore, the axiom (P2) is equivalent to the fact that

$$C_1^-(f,g) = \frac{i}{2}\{f, g\}\,, \tag{4.3}$$

for all $f,\ g \in C^\infty(M)$. Then formula (4.1) for C_1^- is a consequence of the Leibniz rule for the Poisson bracket, and this shows that C_1^+ is a symmetric Hochschild cocycle. Then by [15, Th. 2.15], it is a *Hochschild coboundary*, meaning that there exists a differential operator $D : C^\infty(M) \to C^\infty(M)$ vanishing on constants such that for $f,\ g \in C^\infty(M)$, we have

$$C_1^+(f,g) = D(f)g + fD(g) - D(fg)\,. \tag{4.4}$$

Furthermore, the axiom (P1) implies that the operator $T_k(f) \in \mathrm{End}(H_k)$ is Hermitian for all $k \in \mathbb{N}$ big enough if and only if $f \in C^\infty(M, \mathbb{C})$ is real valued. As the square of a Hermitian operator is Hermitian, the axiom (P3) then shows that C_1^+ is a real-valued bi-differential operator, so that D has real coefficients.

Let us now assume that $C_1^+ \equiv 0$, and consider the bi-differential operators $\hat{C}_1$ and $\hat{C}_2$ defined by interchanging $f, g \in C^\infty(M)$ in axiom (P3) in the following way, as $k \to +\infty$,

$$T_k(g)T_k(g) =: T_k\left(fg + \frac{1}{k}\hat{C}_1(f,g) + \frac{1}{k^2}\hat{C}_2(f,g)\right) + \mathcal{O}(1/k^3)\,. \tag{4.5}$$

Then we have $\hat{C}_1(f,g) = C_1(g,f) = -C_1(f,g)$ and $\hat{C}_2(f,g) = C_2(g,f)$. On the other hand, associativity of composition of operators implies that (4.2) holds for $\hat{C}_1$ and $\hat{C}_2$, one readily checks that $\partial_H C_2 = \partial_H \hat{C}_2$. Therefore, $C_2^- = (C_2 - \hat{C}_2)/2$ is an anti-symmetric Hochschild cocycle, and by [15, Th. 2.15], there exists a 2-form $\alpha \in \Omega^2(M,\mathbb{C})$ so that for all $f, g \in C^\infty(M)$, we have

$$C_2^-(f,g) = \frac{i}{2}\alpha(\operatorname{sgrad} f, \operatorname{sgrad} g)\,. \tag{4.6}$$

Furthermore, by axiom (P1) as above and the fact that the commutator of Hermitian operators is skew-Hermitian, the axiom (P3) implies that the bi-differential operator iC_2^- is real valued, so that α is a real 2-form.

The proofs of Theorem 1.8 and 4.4 are based on a natural operation on quantizations, which we call a *change of variable*. Specifically, given a geometric quantization $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$, and a differential operator $D : C^\infty(M) \to C^\infty(M)$, set

$$T_k^D(f) := T_k\left(f + \frac{1}{k}D\,f\right)\,, \tag{4.7}$$

for all $f \in C^\infty(M)$ and all $k \in \mathbb{N}$. Then one readily checks that the maps $\{T_k^D : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$, satisfy the axioms of Definition 1.6, and that for any $f \in C^\infty(M)$, we have the estimate

$$\left\|T_k(f) - T_k^D(f)\right\|_{op} = \mathcal{O}(1/k)\,, \tag{4.8}$$

as $k \to +\infty$. We will write $C_{1,D}$ and $C_{2,D}$ for the associated bi-differential operators of axiom (P3).

We will use the operation of change of variables to reduce the proof of Theorem 1.8 to a class of remarkable quantizations, described by the following result.

Lemma 4.1. *Assume that (M,ω) satisfies $\dim M = 2$. Then for any geometric quantization $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$, there exists a differential* *operator* $D : C^\infty(M) \to C^\infty(M)$ *vanishing on constants such that the bi-differential operators of axiom (P3) associated with the induced quantization* $\{T_k^D : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$*, satisfy*

$$C^+_{1,D}(f,g) = 0 \qquad \textit{and} \qquad C^-_{2,D}(f,g) = -\frac{i}{2}c\,\{f,g\}\,, \tag{4.9}$$

for all $f,\, g \in C^\infty(M)$*, where* $c \in \mathbb{R}$ *is constant.*

Proof. One readily computes that a change of variable (4.7) associated to a differential operator $D : C^\infty(M) \to C^\infty(M)$ acts on the bi-differential operators C_1^+ and C_2^- via the following formula, for all $f,\, g \in C^\infty(M)$,

$$\begin{aligned} C^+_{1,D}(f,g) &= C^+_1(f,g) + D(f)g + fD(g) - D(fg)\,, \\ C^-_{2,D}(f,g) &= C^-_2(f,g) + \frac{i}{2}\Big(\{D(f),g\} + \{f,D(g)\} - D(\{f,g\})\Big)\,. \end{aligned} \tag{4.10}$$

In particular, formula (4.4) shows that there is an operator D satisfying $C^+_{1,D} \equiv 0$, determined up to the addition of a derivation $\delta : C^\infty(M) \to C^\infty(M)$.

Let now $D : C^\infty(M) \to C^\infty(M)$ be such that that $C^+_{1,D} \equiv 0$, and let $\alpha_D \in \Omega^2(M,\mathbb{R})$ be the two form of formula (4.6) associated with $C^-_{2,D}$. Recall that we assume $\dim M = 2$, so that $H^2(M,\mathbb{R})$ is 1-dimensional, generated by the cohomology class $[\omega]$. Then if we set

$$c := \frac{1}{2\pi}\int_M \alpha_D\,, \tag{4.11}$$

we know that there exists a 1-form $\theta \in \Omega^1(M,\mathbb{R})$ such that

$$\alpha_D = c\,\omega + d\theta\,. \tag{4.12}$$

On the other hand, for all $f,\, g \in C^\infty(M)$, we have by definition

$$\begin{aligned} &d\theta(\operatorname{sgrad} f, \operatorname{sgrad} g) \\ &\qquad = \theta(\operatorname{sgrad}\{f,g\}) - \{\theta(\operatorname{sgrad} f), g\} - \{f, \theta(\operatorname{sgrad} g)\}\,. \end{aligned} \tag{4.13}$$

Then if we consider the derivation $\delta : C^\infty(M) \to C^\infty(M)$ defined for all $f \in C^\infty(M)$ by $\delta f := \theta(\operatorname{sgrad} f)$, formulas (4.10) and (4.12) imply

$$C^-_{2,D+\delta}(f,g) = \frac{i}{2}c\,\omega(\operatorname{sgrad} f, \operatorname{sgrad} g) = -\frac{i}{2}c\,\{f,g\}\,, \tag{4.14}$$

and $C^+_{1,D+\delta} = C^+_{1,D} \equiv 0$. This shows the result. □

Let us end this Section with an existence Theorem, which was already alluded to in Example 1.7.

Theorem 4.2. *[5] Let (M,ω) be a closed symplectic manifold with $[\omega] \in 2\pi H^2(M,\mathbb{Z})$ admitting a complex structure compatible with ω. Then there exists a geometric quantization $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$, such that for all $f \in C^\infty(\mathbb{T}^2,\mathbb{C})$, its $\mathbb{C}$-linear extension satisfies*

$$\|T_k(f)\|_{op} \leq \|f\|_\infty . \tag{4.15}$$

4.2 Proof of Theorem 1.8

Using the estimate (4.8) and Lemma 4.1, we see that it suffices to establish Theorem 1.8 for geometric quantizations for which there is a constant $c \in \mathbb{R}$ such that $C_1^+ \equiv 0$ and $C_2^- = -\frac{i}{2}c\{\cdot,\cdot\}$. All geometric quantizations considered in this Section will thus satisfy this property.

The proof of Theorem 1.8 for the two cases $M = S^2$ and $M = \mathbb{T}^2$ follows the same strategy: we first establish (1.13) for a finite set of functions generating a dense subalgebra of $C^\infty(M)$, and then use the quasi-multiplicativity axiom (P3) in a careful way to extend it to the whole $C^\infty(M)$ by density.

Case of $M = S^2$: We will use the Cartesian coordinate functions u_1, u_2, $u_3 \in C^\infty(S^2)$ of S^2 seen as the unit sphere in $\mathbb{R}^2$. The induced volume form ω is the standard volume form of volume 2π, and these coordinate functions satisfy the commutation relation

$$\{u_j, u_{j+1}\} = -2u_{j+2} , \tag{4.16}$$

for all $j \in \mathbb{Z}/3\mathbb{Z}$. Then given a quantization $\{T_k : C^\infty(S^2) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ with $C_1^+ \equiv 0$ and $C_2^- = -\frac{i}{2}c\{\cdot,\cdot\}$, one readily checks from Definition 1.6 and the commutation relations (4.16) that the assumptions of Theorem 1.2 are satisfied for the constant $c \in \mathbb{R}$ and the operators x_1, x_2, $x_3 \in \mathfrak{su}(H_k)$ defined for all $k \in \mathbb{N}$ and $j \in \mathbb{Z}/3\mathbb{Z}$ by

$$x_j := \frac{ik}{2}\,\frac{k}{k-c}\,T_k(u_j) , \tag{4.17}$$

where u_1, u_2, $u_3 \in C^\infty(S^2)$ are the Cartesian coordinates of $S^2 \subset \mathbb{R}^3$. As the assumption $\limsup_{k\to+\infty} \dim H_k/k < 2$ implies in particular that $\dim H_k <$

$2(k+c)$ for all $k \in \mathbb{N}$, it follows that $c \in \mathbb{Z}$ and that $\dim H_k = k + c$ for all $k \in \mathbb{N}$ big enough, which proves the first statement (1.12).

Furthermore, Theorem 1.2 implies that there exist operators $X_1, X_2, X_3 \in \mathfrak{su}(H_k)$ generating an irreducible representation of $\mathfrak{su}(2)$ such that for all $1 \leq j \leq 3$,

$$\left\| \frac{ik}{2} \frac{k}{k-c} T_k(u_j) - X_j \right\|_{op} = \mathcal{O}(1) \,. \tag{4.18}$$

Now if $\{S_k : C^\infty(S^2) \to \mathcal{L}(H_k)\}_{k \in \mathbb{N}}$ is another quantization with same sequence of Hilbert spaces, we get in the same way operators $\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3 \in \mathfrak{su}(H_k)$ generating an irreducible representation of $\mathfrak{su}(2)$ such that for all $1 \leq j \leq 3$,

$$\left\| \frac{ik}{2} \frac{k}{k-c} S_k(u_j) - \widetilde{X}_j \right\|_{op} = \mathcal{O}(1) \,. \tag{4.19}$$

As any two irreducible representations of $\mathfrak{su}(2)$ with same dimension are isomorphic, formulas (4.18) and (4.19) show that there exist unitary operators $U_k : H_k \to H_k$ for all $k \in \mathbb{N}$ such that for all $j \in \mathbb{Z}/3\mathbb{Z}$,

$$\|T_k(u_j) - U_k^{-1} S_k(u_j) U_k\|_{op} = \mathcal{O}(1/k) \,. \tag{4.20}$$

Set $Q_k := U_k^{-1} S_k U_k$ for all $k \in \mathbb{N}$, and note that by transitivity, it suffices to establish (1.13) when $\{T_k : C^\infty(S^2) \to \mathcal{L}(H_k)\}_{k \in \mathbb{N}}$ is the Berezin-Toeplitz quantization of Theorem 4.2.

Consider the decomposition of $L^2(S^2, \mathbb{C})$ into the direct sum of eigenspaces H_n of the Laplace-Beltrami operator Δ with eigenvalue $2n(n+1)$, for each $n \in \mathbb{N}$. Using for instance [4, Cor. 1.1], we know that for any $N \in \mathbb{N}$, there exists $C_N > 0$ such that for any $n \in \mathbb{N}^*$ and $f \in H_n$, we have

$$\|f\|_{C^N} \leq C_N n^{2N} \|f\|_{L^2} \,. \tag{4.21}$$

Recall on the other hand that for any $n \in \mathbb{N}$, the eigenspace H_n is isomorphic to the irreducible $SO(3)$-representations of highest weight $n \in \mathbb{N}$ with respect to $S^1 \subset SO(3)$ rotating along the u_3-axis, and write $f_n \in H_n$ for the unit highest weight vectors. Via the identification with *spherical harmonics* and following e.g. [1, Ex. 15.4.1, § 15.5], we have the following recursion formula in $n \in \mathbb{N}$,

$$\begin{aligned} f_{n+1} &= \sqrt{\frac{2n+3}{2n+2}} f_1 \, f_n \,, \\ f_1 &= -(u_1 + i u_2) \,. \end{aligned} \tag{4.22}$$

Let us prove by induction that there exists constants $\alpha > 0$ and $M \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and all $k \in \mathbb{N}$, we have

$$\|T_k(f_n) - Q_k(f_n)\|_{op} \leq \alpha \frac{n^M}{k}\,, \tag{4.23}$$

The case $n = 1$ readily follows from (4.20) and formula (4.22) for f_1. On the other hand, axioms (P1), (P3) and the estimate (4.21) give constants $\alpha_0 > 0$ and $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} \|Q_k(f_1\, f_n) - Q_k(f_1)Q_k(f_n)\|_{op} &\leq \frac{\alpha_0}{k}\|f_1\|_{C^N} n^{2N}\,, \\ \|Q_k(f_n)\|_{op} &\leq \alpha_0\, n^{2N}\,, \end{aligned} \tag{4.24}$$

and the same holds for $\{T_k : C^\infty(S^2) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$. Let now $n \in \mathbb{N}$ be such that (4.23) and holds, and recall by assumption that $\|T_k(f)\|_{op} \leq \|f\|_\infty$ for all $f \in C^\infty(S^2, \mathbb{C})$ and $k \in \mathbb{N}$. Then using the sub-multiplicativity of the operator norm, we get that for any $f \in C^\infty(S^2)$,

$$\begin{aligned} &\|T_k(f_1\, f_n) - Q_k(f_1\, f_n)\|_{op} \\ &\quad \leq \|T_k(f_1)T_k(f_n) - Q_k(f_1)Q_k(f_n)\|_{op} + 2\frac{\alpha_0}{k}\|f_1\|_{C^N} n^{2N} \\ &\quad \leq \|T_k(f_1)(T_k(f_n) - Q_k(f_n))\|_{op} \\ &\qquad + \|(T_k(f_1) - Q_k(f_1))Q_k(f_n)\|_{op} + 2\frac{\alpha_0}{k}\|f_1\|_{C^N} n^{2N} \\ &\quad \leq \frac{\alpha}{k}\|f_1\|_\infty n^M + \frac{C\alpha_0}{k} n^{2N} + 2\frac{\alpha_0}{k}\|f_1\|_{C^N} n^{2N}\,, \end{aligned} \tag{4.25}$$

where $C > 0$ comes from the estimate (4.20) and formula (4.22) for f_1. As $\|f_1\|_\infty = \max_{x\in S^2} |u_1 + iu_2| = 1$, we can choose $\alpha = \alpha_0(C + 2\|f_1\|_{C^N})$ and $M = 2N + 1$ in (4.23) to get

$$\begin{aligned} \frac{\alpha}{k}\sqrt{\frac{2n+3}{2n+2}}(n^M + n^{2N}) &\leq \frac{\alpha}{k}(n+1)(n^{M-1} + n^{2N-1}) \\ &\leq \frac{\alpha}{k}(n+1)^M\,, \end{aligned} \tag{4.26}$$

where we used the fact that $n\sqrt{\frac{2n+3}{2n+2}} \leq n+1$ for all $n \in \mathbb{N}$. Using (4.22), this implies (4.23) with n replaced by $n+1$, and thus for all $n \in \mathbb{N}$ by induction.

Let us now establish the estimate (4.23) for all functions in H_n, for each $n \in \mathbb{N}$. First, by definition of the action of $SO(3)$ on the unit sphere S^2, we see that (4.20) implies the existence of a constant $C > 0$ such that for any $g \in SO(3)$, any $j \in \mathbb{Z}/3\mathbb{Z}$ and all $k \in \mathbb{N}$, we have

$$\|T_k(g^*u_j) - Q_k(g^*u_j)\|_{op} \leq C/k\,. \tag{4.27}$$

Note on the other hand that for any $g \in SO(3)$, the functions $g^*f_n \in C^\infty(S^2)$, $n \in \mathbb{N}$, are again highest weight vectors with respect to $S^1 \subset SO(3)$ rotating along the g^*u_3-axis. We can then repeat the reasoning above replacing f_n by g^*f_n for all $n \in \mathbb{N}$ to get

$$\|T_k(g^*f_n) - Q_k(g^*f_n)\|_{op} \leq \alpha\frac{n^M}{k}\,, \tag{4.28}$$

for any $g \in SO(3)$, with same constants $\alpha > 0$ and $M \in \mathbb{N}$. Recall on the other hand that the standard volume form ω on S^2 is the pushforward of the Haar measure on $SO(3)$. Then following e.g. [7, III.3.3.a], for any $f \in H_n$ we can consider its *coherent state decomposition*

$$f = \frac{n+1}{2\pi}\int_{S^2}\langle f, g^*f_n\rangle_{L^2}\; g^*f_n\;\omega_{[g]}\,, \tag{4.29}$$

where $g \in SO(3)$ is any representative of $[g] \in S^2 \simeq SO(3)/S^1$. Then using (4.28) and Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&\|T_k(f) - Q_k(f)\|_{op} \\
&\qquad \leq \frac{n+1}{2\pi}\int_{S^2}|\langle f, g^*f_n\rangle_{L^2}|\;\;\|T_k(g^*f_n) - Q_k(g^*f_n)\|_{op}\;\omega_{[g]} \\
&\qquad \leq \alpha\|f\|_{L^2}\frac{(n+1)n^M}{k}\,.
\end{aligned} \tag{4.30}$$

Take now any $f \in C^\infty(S^2)$, and consider its spectral decomposition into the eigenspaces of Δ, so that $f = \sum_{n\in\mathbb{N}} a_n\,\varphi_n$, with $\varphi_n \in H_n$ and $\|\varphi_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$. Since f is smooth, the sequence $(a_n)_{n\in\mathbb{N}}$ decays faster than any power of n, so that using (4.30), there exists $C' > 0$ such that

$$\|T_k(f) - Q_k(f)\|_{op} \leq \alpha\sum_{n\in\mathbb{N}}\frac{(n+1)n^M}{k}a_n \leq \frac{C'}{k}\,. \tag{4.31}$$

This shows formula (1.13) in the case of $M = S^2$.

Case of $M = \mathbb{T}^2$: Write $(q_1, q_2) \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ for the standard coordinates, so that the standard volume form of volume 2π writes $\omega = 2\pi\, dq_1 \wedge dq_2$. We will use the functions u_1, $u_2 \in C^\infty(\mathbb{T}^2, \mathbb{C})$ defined for any $q := (q_1, q_2) \in \mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ and $j = 1, 2$ by

$$u_j(q) := e^{2\pi i q_j}\,, \tag{4.32}$$

which satisfy the commutation relation

$$\{u_1, u_2\} = 2\pi u_1 u_2\,. \tag{4.33}$$

Following for instance in [3, § 2], we consider the *Moyal-Weyl star product* over $(C^\infty(\mathbb{T}^2), \{\cdot,\cdot\})$, defined as in (1.10) with coefficients $\widetilde{C}_1$ and $\widetilde{C}_2$ satisfying $\widetilde{C}_1^+ = \widetilde{C}_2^- = 0$ and for all f, $g \in C^\infty(\mathbb{T}^2)$,

$$\widetilde{C}_2^+(f,g) = -\frac{1}{32\pi^2}\left(\frac{\partial^2}{\partial q_1^2} f\,\frac{\partial^2}{\partial q_2^2} g - 2\frac{\partial^2}{\partial q_1 \partial q_2} f\,\frac{\partial^2}{\partial q_1 \partial q_2} g + \frac{\partial^2}{\partial q_2^2} f\,\frac{\partial^2}{\partial q_1^2} g\right)\,. \tag{4.34}$$

Then given a geometric quantization $\{T_k : C^\infty(\mathbb{T}^2) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ with $C_1^+ \equiv 0$ and $C_2^- = -\frac{i}{2}c\,\{\cdot,\cdot\}$, we get that $C_1 = \widetilde{C}_1$ via formula (4.3), and using (4.2), we get that $C_2 - \widetilde{C}_2$ is a Hochschild cocycle (4.1). On the other hand, we have $C_2^- - \widetilde{C}_2^- = -\frac{i}{2}c\,\{\cdot,\cdot\}$, which also satisfies (4.1), and we thus get $C_2^+ - \widetilde{C}_2^+$ is a symmetric Hochschild cocycle, hence a coboundary by [15, Th. 2.15]. As in formula (3.11), this means that there exists a differential operator $D_2 : C^\infty(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2)$ vanishing on constants such that

$$\widetilde{C}_2^+(f,g) = C_2^+(f,g) + D_2(f)g + fD_2(g) - D_2(fg)\,. \tag{4.35}$$

Consider the following change of variables at second order in $1/k$, for all $f \in C^\infty(\mathbb{T}^2)$,

$$T_k^{D_2}(f) := T_k\left(f + \frac{1}{k^2}D_2(f)\right)\,. \tag{4.36}$$

This again defines a geometric quantization in the sense of Definition 1.6, with associated bi-differential operators C_{1,D_2} and C_{2,D_2} of axiom (P3) satisfying $C_{1,D_2}^+ \equiv 0$ and $C_{2,D_2}^- = -\frac{i}{2}c\,\{\cdot,\cdot\}$, while formula (4.35) implies $C_{2,D_2}^+ = \widetilde{C}_2^+$. Note also that $T_k^{D_2}(u_j)^* = T_k^{D_2}(u_j^{-1})$ for each $j = 1, 2$ by Definition 1.6 and formula (4.32). Then using formula (4.34), we get as $k \to +\infty$,

$$\begin{aligned} T_k^{D_2}(u_j)T_k^{D_2}(u_j)^* &= \mathbb{1} + \frac{i}{2}\left(\frac{1}{k} - \frac{c}{k^2}\right)T_k^{D_2}(\{u_j, u_j^{-1}\}) + \mathcal{O}(1/k^3) \\ &= \mathbb{1} + \mathcal{O}(1/k^3) \end{aligned}\,, \tag{4.37}$$

and

$$
\begin{aligned}
T_k^{D_2}(u_1)T_k^{D_2}(u_2) = T_k^{D_2}(u_1u_2) &+ \frac{i}{2}\left(\frac{1}{k} - \frac{c}{k^2}\right)T_k^{D_2}(\{u_1,u_2\}) \\
&- \frac{1}{32\pi^2k^2}T_k^{D_2}\left(\frac{\partial^2}{\partial q_1^2}u_1\frac{\partial^2}{\partial q_2^2}u_2\right) + O(1/k^3) \\
= T_k^{D_2}(u_1u_2) + \frac{i}{2}\frac{2\pi}{k+c}&T_k^{D_2}(u_1u_2) - \frac{(2\pi)^2}{8k^2}T_k^{D_2}(u_1u_2) + O(1/k^3) \\
&= e^{2\pi i/2(k+c)}T_k^{D_2}(u_1u_2) + O(1/k^3)\,, \quad (4.38)
\end{aligned}
$$

while in the same way,

$$
T_k^{D_2}(u_2)T_k^{D_2}(u_1) = e^{-2\pi i/2(k+c)}T_k^{D_2}(u_1u_2) + O(1/k^3)\,. \tag{4.39}
$$

We then see that the operators $x_j := T_k^{D_2}(u_j)$ for all $j = 1, 2$ and $k \in \mathbb{N}$, satisfy the assumptions of Theorem 1.5 for the constant $c \in \mathbb{R}$ as above. As the assumption $\limsup_{k\to+\infty} \dim H_k/k < 2$ implies in particular that $\dim H_k < 2(k+c)$ for all $k \in \mathbb{N}$, it follows that $c \in \mathbb{Z}$ and that $\dim H_k = k+c$ for all $k \in \mathbb{N}$ big enough, which proves (1.12). Furthemore, Theorem 1.5 and formula (4.36) imply that there exist unitary operators $X_1, X_2 \in \mathrm{End}(H_k)$ satisfying $X_1X_2 = e^{2\pi i/(k+c)}X_2X_1$ and not preserving any non-trivial proper subspace, such that

$$
\|T_k(u_j) - X_j\|_{op} = \mathcal{O}(1/k) \quad \text{for all} \quad j = 1,\, 2\,. \tag{4.40}
$$

Note that the explicit formula (2.53) shows that for any two such pairs $X_1, X_2 \in \mathrm{End}(H_k)$ and $\widetilde{X}_1, \widetilde{X}_2 \in \mathrm{End}(H_k)$, there exists a unitary operator $U : H_k \to H_k$ and $p := (p_1, p_2) \in \mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ such that for each $j = 1, 2$, we have

$$
\widetilde{X}_j = e^{2\pi i p_j}U^{-1}X_jU\,. \tag{4.41}
$$

Setting $m_j := \lfloor (k+c)p_j \rfloor \in \mathbb{N}$ for each $j = 1, 2$, and considering the unitary operator $U_{m_1,m_2} := X_1^{-m_2}X_2^{m_1} \in \mathrm{End}(H_k)$, we get the following estimates in operator norm as $k \to +\infty$, for all $p = (p_1, p_2) \in \mathbb{T}^2$ and all unitary operators $X_1, X_2 \in \mathrm{End}(H_k)$ satisfying the above commutation relations,

$$
U_{m_1,m_2}X_jU_{m_1,m_2}^{-1} = e^{-2\pi i m_j/(k+c)}X_j = e^{-2\pi i p_j}X_j + \mathcal{O}(1/k)\,. \tag{4.42}
$$

Thus for any two such pairs of sequences $X_1, X_2 \in \mathrm{End}(H_k)$, $k \in \mathbb{N}$, and $\widetilde{X}_1, \widetilde{X}_2 \in \mathrm{End}(H_k)$, $k \in \mathbb{N}$, we get a sequence of unitary operators $U_k : H_k \to$

H_k, $k \in \mathbb{N}$, such that

$$\widetilde{X}_j = U_k^{-1} X_j U_k + \mathcal{O}(1/k)\,. \tag{4.43}$$

Then if we have two quantizations $\{T_k,\, Q_k : C^\infty(\mathbb{T}^2) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ with same sequence of Hilbert spaces satisfying $\dim H_k = k + c$ for all $k \in \mathbb{N}$ big enough, they both satisfy (1.8) for two different pairs $X_1,\, X_2 \in \mathrm{End}(H_k)$ and $\widetilde{X}_1,\, \widetilde{X}_2 \in \mathrm{End}(H_k)$, and formula (4.43) shows that there exists unitary operators $U_k : H_k \to H_k$ for all $k \in \mathbb{N}$ such that for each $j = 1,\, 2$,

$$\|U_k^{-1} Q_k(u_j) U_k - T_k(u_j)\|_{op} = \mathcal{O}(1/k)\,. \tag{4.44}$$

Now by transitivity as above, it suffices to establish (1.13) when $\{T_k : C^\infty(\mathbb{T}^2) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$, is the Berezin-Toeplitz quantization of Theorem 4.2. Then by a straightforward adaptation of the computation (4.25) with f_1 replaced by u_1, u_2 respectively and f_n replaced by $u_1^n u_2^m$, we get by induction on $n,\, m \in \mathbb{Z}$ that there exist constants $\alpha > 0$ and $M \in \mathbb{N}$, depending only on the quantizations, such that

$$\|T_k(u_1^n u_2^m) - Q_k(u_1^n u_2^m)\|_{op} \leq \alpha \frac{(|n| + |m|)^M}{k}\,. \tag{4.45}$$

Now for any $f \in C^\infty(\mathbb{T}^2)$, consider its Fourier expansion

$$f = \sum_{m,\, n \in \mathbb{Z}} a_{m,n} u_1^n u_2^m\,. \tag{4.46}$$

Since f is smooth, the coefficients $a_{n,m}$, $n,\, m \in \mathbb{Z}$, decay faster than any polynomial in $|n|$, $|m|$. Using the estimate (4.45) in the same way as with (4.31), this shows formula (1.13) in the case of $M = \mathbb{T}^2$, and concludes the proof of Theorem 1.8.

4.3 Traces of quantizations

Note that in the previous section, we showed in particular that for geometric quantizations of $M = S^2$ or $\mathbb{T}^2$, the constant $c \in \mathbb{R}$ appearing in Lemma 4.1 is an integer, uniquely determined by the condition $\dim H_k = k + c$ for all $k \in \mathbb{N}$ big enough. This fact can be refined for geometric quantizations satisfying the following additional axiom.

Definition 4.3. A geometric quantization $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ of a closed symplectic manifold (M, ω) of dimension $\dim M = 2d$ is said to satisfy the *trace axiom* if there exists a function $R \in C^\infty(S^2)$ such that for all $f \in C^\infty(S^2)$, we have

$$\operatorname{tr} T_k(f) = \left(\frac{k}{2\pi}\right)^d \int_M f\, R_k\, \frac{\omega^d}{d!}\,, \tag{4.47}$$

for a sequence of functions $R_k \in C^\infty(M)$ satisfying the following estimate as $k \to +\infty$,

$$R_k = 1 + \frac{1}{k} R + \mathcal{O}(1/k^2)\,.$$

We then have the following refinement of Lemma 4.1, relating this trace with the coefficient C_2^-.

Theorem 4.4. *Let $M = S^2$ or $\mathbb{T}^2$ be endowed with the standard volume form ω of volume 2π. Then if $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ is a geometric quantization with $C_1^+ \equiv 0$ satisfying the trace axiom of Definition 4.3, we have for all f, $g \in C^\infty(M)$,*

$$C_2^-(f,g) = -\frac{i}{2} R\,\{f,g\}\,. \tag{4.48}$$

Proof. Let $T_k : C^\infty(M) \to \mathcal{L}(H_k)$, $k \in \mathbb{N}$, be a geometric quantization with $C_1^+ \equiv 0$ satisfying the trace axiom of Definition (4.3), and recall the form $\alpha \in \Omega^2(M,\mathbb{R})$ of formula (4.6). Let $c \in \mathbb{R}$ and $\theta \in \Omega^1(M,\mathbb{R})$ be such that $\alpha = c\,\omega + d\theta$, as in formula (4.12), and write

$$d\theta =: R_\theta\, \omega\,, \tag{4.49}$$

with $R_\theta \in C^\infty(M)$. Considering the change of variable (4.7) induced by the derivation $\delta : C^\infty(M) \to C^\infty(M)$ defined by $\delta f := \theta(\operatorname{sgrad} f)$, we compute

$$\int_M \delta f\, \omega = -\int_M f\, d\theta = -\int_M R_\theta\, f\, \omega\,. \tag{4.50}$$

Then one readily computes that the quantization $\{T_k^\delta : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ induced by δ as in (4.7) also satisfies the trace axiom of Definition 4.3, where the function $R \in C^\infty(M)$ is replaced by the function $R_\delta := R - R_\theta$. On the other hand, we know from the proof of Lemma 4.1 that $C_{1,\delta}^+ = C_1^+ \equiv 0$ and

$C^-_{2,\delta} = -\frac{i}{2}c\{\cdot,\cdot\}$, and from the proof of Theorem 1.8 above that $c \in \mathbb{R}$ is an integer satisfying $\dim H_k = k + c$ for all $k \in \mathbb{N}$ big enough. Applying formula (4.47) to $f = 1$ and using that $T_k^\delta(1) = \mathbb{1}$, we get

$$\frac{1}{2\pi}\int_M R_\delta\,\omega = c\,. \tag{4.51}$$

On the other hand, using the axioms (P2) and (P3), we get for any f, $g \in C^\infty(M)$ that as $k \to +\infty$,

$$\begin{aligned} i\left(1 - \frac{c}{k}\right)\operatorname{tr} T_k^\delta(\{f,g\}) &= k\operatorname{tr}\left([T_k^\delta(f), T_k^\delta(g)] + \mathcal{O}(1/k^3)\right) \\ &= \mathcal{O}(1/k)\,. \end{aligned} \tag{4.52}$$

Now as every function with zero mean can be written as a sum of Poisson brackets (see e.g. [2, Theorem 1.4.3]), we get that

$$\int_M f\,\omega = 0 \quad \text{implies} \quad \operatorname{tr} T_k^\delta(f) = \mathcal{O}(1/k) \text{ as } k \to +\infty\,. \tag{4.53}$$

Using formula (4.47) again, we see that this is possible if and only R_δ is constant, equal to $c \in \mathbb{Z}$ by formula (4.51). We thus have $R = c + R_\theta$, and by formulas (4.13) and (4.49), we get

$$C_2^-(f,g) = -\frac{i}{2}c\{f,g\} - \frac{i}{2}R_\theta\{f,g\} = -\frac{i}{2}R\{f,g\}\,. \tag{4.54}$$

This gives the result. □

Together with Theorem 1.5, Theorem 4.4 implies the following extension of Theorem 1.8 in a special case. Denote by $\tau_p : \mathbb{T}^2 \to \mathbb{T}^2$ the translation by an element $p \in \mathbb{T}^2$.

Theorem 4.5. *Let $\{Q_k,\, T_k : C^\infty(\mathbb{T}^2) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ be two geometric quantizations of the torus satisfying the trace axiom of Definition 4.3, with the same sequence of Hilbert spaces, same C_1^+, and same $R \in C^\infty(\mathbb{T}^2)$. Assume furthermore that the function R is constant, and that the bi-differential operator C_1^+ is translation invariant. Then there exist a sequence $\{U_k : H_k \to H_k\}_{k\in\mathbb{N}}$ of unitary operators and a sequence $\{p_k \in \mathbb{T}^2\}_{k\in\mathbb{N}}$ of points in $\mathbb{T}^2$ such that for any $f \in C^\infty(\mathbb{T}^2)$, we have as $k \to +\infty$,*

$$\|U_k^{-1}Q_k(\tau_{p_k}^* f)U_k - T_k(f)\|_{op} = \mathcal{O}(1/k^{3/2})\,. \tag{4.55}$$

Proof. First note that as C_1^+ is translation invariant, the differential operator $D : C^\infty(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2)$ appearing in formula (4.4) can be chosen to be translation invariant as well. Consider the common change of variable defined for all $f \in C^\infty(\mathbb{T}^2)$ and $k \in \mathbb{N}$ by

$$Q_k^D(f) := Q_k\left(f + \frac{1}{k}D(f)\right) \quad \text{and} \quad T_k^D(f) := T_k\left(f + \frac{1}{k}D(f)\right) , \quad (4.56)$$

so that $C_{1,D}^+ \equiv 0$. Since both D and ω are translation invariant, and D vanishes on constants, we have

$$\int_{\mathbb{T}^2} D(f)\,\omega = 0 , \quad (4.57)$$

for all $f \in C^\infty(\mathbb{T}^2)$. We then see that the quantizations (4.56) also satisfy the trace axiom of Definition 4.3 with function $R \in C^\infty(\mathbb{T}^2)$ unchanged, and by Theorem 4.4, they have same $C_{2,D}^-$, given for all $f, g \in C^\infty(\mathbb{T}^2)$ by the formula

$$C_{2,D}^-(f,g) = -\frac{i}{2}R\,\{f,g\} . \quad (4.58)$$

Furthermore, the trace axiom of Definition 4.3 implies in particular that $\limsup_{k\to+\infty} \dim H_k/k < 2$. As R is constant by assumption, we can then follow the proof of Theorem 1.8 in Section 4.2 in the torus case $\mathbb{T}^2$, replacing the quantizations $\{Q_k, T_k : C^\infty(\mathbb{T}^2) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ by the quantizations $\{Q_k^D, T_k^D : C^\infty(\mathbb{T}^2) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ constructed above. Using the full strength of Theorem 1.5, we get unitary operators $X_1, X_2 \in \operatorname{End}(H_k)$ satisfying $X_1X_2 = e^{2\pi i/(k+c)}X_2X_1$ such that the following analogue of formula (4.40) holds,

$$\|T_k^D(u_j) - X_j\|_{op} = \mathcal{O}(1/k^{3/2}) \quad \text{for all} \quad j = 1, 2 . \quad (4.59)$$

Furthermore, the same holds for Q_k^D with operators $\widetilde{X}_1, \widetilde{X}_2 \in \operatorname{End}(H_k)$ such that

$$\widetilde{X}_j = e^{2\pi i p_j} U^{-1} X_j U \quad \text{for all} \quad j = 1, 2 , \quad (4.60)$$

for a unitary operator $U : H_k \to H_k$ and $p := (p_1, p_2) \in \mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$. On the other hand, note that by definition (4.32) of $u_j \in C^\infty(\mathbb{T}^2, \mathbb{C})$ for all $j = 1, 2$, if $\tau_p : \mathbb{T}^2 \to \mathbb{T}^2$ is the translation operator by $p \in \mathbb{T}^2$, then we have

$$\tau_p^* u_j = e^{2\pi i p_j} u_j . \quad (4.61)$$

Using now the commutation relation (4.33) and the fact that $C^+_{1,D} \equiv 0$, we get from axiom (P3) a constant $\alpha > 0$ and a constant $N \in \mathbb{N}$ such that for any $m, n \in \mathbb{Z}$, we have

$$\begin{aligned} \left\| T_k^D(u_1) T_k^D(u_1^n u_2^m) - \left(1 + \frac{2\pi i n}{k}\right) T_k^D(u_1^{n+1} u_2^m) \right\|_{op} \leq \frac{\alpha}{k^2} (|n| + |m|)^N , \\ \left\| T_k^D(u_2) T_k^D(u_1^n u_2^m) - \left(1 + \frac{2\pi i m}{k}\right) T_k^D(u_1^n u_2^{m+1}) \right\|_{op} \leq \frac{\alpha}{k^2} (|n| + |m|)^N . \end{aligned} \tag{4.62}$$

Using this estimate and through a traightforward adaptation of the proof in Section 4.2 for $M = \mathbb{T}^2$, we then get a sequence of unitary operators $\{U_k : H_k \to H_k\}_{k\in\mathbb{N}}$, $k \in \mathbb{N}$ and a sequence of points $\{p_k \in \mathbb{T}^2\}_{k\in\mathbb{N}}$, such that for any $f \in C^\infty(S^2)$, we have the following estimate as $k \to +\infty$,

$$\| U_k^{-1} Q_k^D(\tau_{p_k}^* f) U_k - T_k^D(f) \|_{op} = \mathcal{O}(1/k^{3/2}) , \tag{4.63}$$

where $\tau_p : \mathbb{T}^2 \to \mathbb{T}^2$ denotes the translation by $p \in \mathbb{T}^2$. As the common change of variable (4.56) is invariant by translation, this readily implies the result. □

Theorem 4.4 is of specific interest in the theory of *deformation quantization* of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$. To see this, consider the following extension of axiom (P3).

Definition 4.6. A geometric quantization $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ of a closed symplectic manifold (M, ω) is said to satisfy the *star product axiom* if there exists a collection of bi-differential operators C_j, $j \in \mathbb{N}$, such that for all $m \in \mathbb{N}$ and all $f, g \in C^\infty(M)$,

$$T_k(f) T_k(g) = T_k \left(fg + \sum_{j=1}^{m-1} \frac{1}{k^j} C_j(f, g) \right) + \mathcal{O}(1/k^m) . \tag{4.64}$$

The name for this axiom is justified by the fact that, together with the other axioms of Definition 1.6, this induces a *differential star product* $*$ on the ring of formal power series $C^\infty(M, \mathbb{C})[[\hbar]]$, with formal parameter $\hbar$. Specifically, the formula

$$f * g := fg + \sum_{j=1}^{\infty} \hbar^j C_j(f, g) , \tag{4.65}$$

for all $f,\, g \in C^\infty(M)$, defines an associative unital $\mathbb{C}[[\hbar]]$-linear product $*$ on $C^\infty(M,\mathbb{C})[[\hbar]]$ satisfying $f*g-g*f = i\hbar\{f,g\}+\mathcal{O}(\hbar^2)$. Setting $\hbar = 1/k$, we see that (4.6) reads formally as the *star product axiom*

$$T_k(f)T_k(g) = T_k(f*g)\,, \tag{4.66}$$

where this equality is understood as an asymptotic expansion with respect to the operator norm.

Working with formal power series in $\hbar$, one can extend the notions (4.7) and (4.36) of a change of variable over any subset $U \subset M$ as a map $A : C^\infty(U,\mathbb{C})[[\hbar]] \longrightarrow C^\infty(U,\mathbb{C})[[\hbar]]$ satisfying $A(1) = 1$ and

$$A(f) := f + \sum_{j=1}^{+\infty} \hbar^j D_j\, f\,, \tag{4.67}$$

for all compactly supported $f \in C^\infty(U)$, where D_j are differential operators for all $j \in \mathbb{N}$. This acts on a star product $*$ via the formula

$$f *_A g := A^{-1}(A(f) * A(g))\,, \tag{4.68}$$

where $*_A$ is defined on compactly supported functions $f,\, g \in C^\infty(U,\mathbb{C})$. In the theory of deformation quantization, this is also called a *star-equivalence*. For change of variables of the form $A(f) = f + \hbar D\, f$ for any $f \in C^\infty(M)$, one readily checks that $*_A$ is the star product (4.66) associated to the geometric quantization $\{T_k^D : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ of (4.7).

Following [25, § 1, p.229] (see also [19, § 2, p.220]), one can define the *canonical trace* of a differential star product $\star$ over a closed symplectic manifold (M,ω) of dimension $\dim M = 2d$ as the map $\mathrm{tr}_\hbar : C^\infty(M)[[\hbar]] \to \mathbb{C}[[\hbar]]$ such that for any $f \in C^\infty(M)$ supported over a contractible Darboux chart $U \subset M$, we have

$$\mathrm{tr}_\hbar(f) = (2\pi\hbar)^{-d} \int_X A_U(f)\, \frac{\omega^d}{d!}\,, \tag{4.69}$$

where $A_U : C^\infty(U)[[\hbar]] \to C^\infty(U)[[\hbar]]$ is a change of variable making $\star$ equal to the usual Moyal-Weyl star product over $\mathbb{R}^{2d}$ in these Darboux charts. We will not need the full definition of the Moyal-Weyl star product, but only that it satisfies $C_1^+ = C_2^- = 0$. The following result is then a consequence of Theorem 4.4.

Corollary 4.7. *Let $M = S^2$ or $\mathbb{T}^2$ be endowed with the standard volume form ω of the total area 2π. Let $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ be a geometric quantization satisfying the trace axiom of Definition 4.3 and the star product axiom of Definition 4.6. Then for all $f \in C^\infty(S^2)$, we have the asymptotic expansion*

$$\operatorname{tr} T_k(f) = \operatorname{tr}_\hbar(f) + \mathcal{O}(1/k) \,,$$

as $k = 1/\hbar \to +\infty$.

Proof. Take $f \in C^\infty(M)$ to be compactly supported in a Darboux chart $U \subset M$, and let $A_U : C^\infty(U)[[\hbar]] \to C^\infty(U)[[\hbar]]$ be a local change of variable making the induced star product (4.65) equal to the Moyal-Weyl star product. Let us write

$$A_U(f) = f + \hbar\, D_U\, f + O(\hbar^2)\,, \tag{4.70}$$

and write $\widetilde{C}_1$ and $\widetilde{C}_2$ for the bi-differential operators of (4.65) associated with the star product $*_{A_U}$ over U. Note that terms of order $\hbar^2$ and more do not affect $\widetilde{C}_1^+$ and $\widetilde{C}_2^-$, and by formula (4.10), the condition $\widetilde{C}_1^+ \equiv 0$ determines $D_U : C^\infty(U) \to C^\infty(U)$ up to a derivation. In particular, by the trace axiom (4.47), one sees that both the usual trace and the canonical trace change the same way under a change of variable of the form (4.7). By Lemma 4.1, it suffices to show the result for quantizations which already satisfy $C_1^+ \equiv 0$.

Let then $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ be a geometric quantization with $C_1^+ \equiv 0$ and satisfying the trace axiom (4.47), so that we are under the hypotheses of Theorem 4.4. Then by formula (4.10), the condition $\widetilde{C}_1^+ \equiv 0$ implies that $D_U : C^\infty(U) \to C^\infty(U)$ has to be a derivation in that case. Furthermore, formulas (4.10) and (4.13) show that in order to also have $\widetilde{C}_2^- \equiv 0$, this derivation has to be of the form $D_U\, f := -\theta(\text{sgrad } f)$ for all compactly supported $f \in C^\infty(U)$, where $\theta \in \Omega^1(M, \mathbb{R})$ satisfies

$$C_2^-(f, g) = \frac{i}{2} d\theta(\text{sgrad} f, \text{sgrad} g)\,, \tag{4.71}$$

for all compactly supported f, $g \in C^\infty(U)$. Note that this is compatible with formula (4.6), as all 2-forms over a contractible open set $U \subset M$ are exact. Then by definition (4.69) of the canonical trace, for all $f \in C^\infty(M)$ with compact support in $U \subset X$, we then have

$$\begin{aligned} \operatorname{tr}_\hbar(f) &= \frac{1}{2\pi\hbar}\int_X (f - \hbar\,\theta(\mathrm{sgrad} f))\,\omega + O(\hbar) \\ &= \frac{1}{2\pi\hbar}\int_X (f + \hbar\, f R_U)\,\omega + O(\hbar)\,, \end{aligned} \tag{4.72}$$

where $R_U \in C^\infty(U)$ is defined by the formula

$$d\theta =: R_U\,\omega|_U\,. \tag{4.73}$$

Therefore, by formula (4.71), for all compactly supported $f,\, g \in C^\infty(U)$, we have

$$C_2^-(f,g) = -\frac{i}{2}\,R_U\,\{f,g\}\,. \tag{4.74}$$

By Theorem 4.4, the trace $\operatorname{tr} T_k(f)$ is given by the last term in formula (4.72), and hence coincides with the canonical trace $\operatorname{tr}_\hbar(f)$ up to $\mathcal{O}(1/k)$. This completes the proof of the corollary. □

Corollary 4.7 naturally leads to the following conjecture.

Conjecture 4.8. *Let $\{T_k : C^\infty(M) \to \mathcal{L}(H_k)\}_{k\in\mathbb{N}}$ be a geometric quantization of a closed symplectic manifold (M,ω) satisfying the trace axiom of Definition 4.3 and the star product axiom of Definition 4.6. Then for all $f \in C^\infty(M)$ and $m \in \mathbb{N}$, we have the asymptotic expansion*

$$\operatorname{tr} T_k(f) = \operatorname{tr}_\hbar(f) + O(1/k^m)\,,$$

as $k = 1/\hbar \to +\infty$.

The trace axiom of Definition 4.3 is a basic property of Berezin-Toeplitz quantizations of closed Kähler manifolds, and the fact that these quantizations satisfy the star product axiom of Definition 4.6 has been shown by Schlichenmaier in [29]. Then Conjecture 4.8 for Berezin-Toeplitz quantizations of closed Kähler manifolds has been established by Hawkins in [17, Cor. 10.5].

Remark 4.9. As explained for instance in [15, § 6], there exists a notion of characteristic class for differential star-products $*$ over symplectic manifolds, which has been introduced by Deligne in [11] as an element $c(\star)$ of the affine space $\hbar^{-1}[\omega] + H^2(M,\mathbb{R})[[\hbar]]$. By the work of Fedosov [12] and Nest and

Tsygan [25, 26], this class is known to classify star-products up to *star-equivalence* (4.68). Then we have the relation

$$c(\star) = \hbar^{-1}[\omega] + c\,[\omega] + \mathcal{O}(\hbar)\,, \tag{4.75}$$

where $c \in \mathbb{R}$ is the constant produced from $T_k : C^\infty(M) \to \mathcal{L}(H_k)$, $k \in \mathbb{N}$ by Lemma 4.1. Then for geometric quantizations satisfying star product axiom (4.66), the proof of Theorem 1.8 computes this constant to be an integer via the formula $\dim H_k = k + c$ for all $k \in \mathbb{N}$. The Deligne-Fedosov class of the standard Berezin-Toeplitz quantizations of closed Kähler manifolds has been computed by Hawkins in [17, Th. 10.6] and Karabegov and Schlichenmaier in [20].

Acknowledgement. L.P. thanks University of Chicago, where a part of this paper was written, for hospitality and an excellent research atmosphere. We thank D.Treschev for a useful comment, and O. Shabtai for an attentive reading of the manuscript and pointing out a number of mistakes.

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