ON PETERSSON NORMS OF GENERIC CUSP FORMS AND SPECIAL VALUES OF ADJOINT L-FUNCTIONS FOR GSp_4

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Abstract. We prove an explicit formula for the Petersson norms of some normalized generic cuspidal newforms on GSp_4 whose archimedean components belong to either discrete series representations or spherical principal series representations. Our formula expresses the Petersson norms in terms of special values of adjoint L-functions and some elementary constants depending only on local representations.

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1. Introduction

In the study of automorphic forms, it is natural and important to measure their size. For example, let $f \in S_\kappa(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and consider its Petersson norm

$$\langle f, f\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} |f(\tau)|^2 \,\mathrm{Im}(\tau)^{\kappa-2}\, d\tau.$$

Then

$$\langle f, f\rangle = 2^{-\kappa} L(1, f, \mathrm{Ad}),$$

where $L(s, f, \mathrm{Ad})$ is the (completed) adjoint L-function of f, and this formula has many applications in analytic number theory and arithmetic geometry.

More generally, let k be a number field with adele ring $\mathbb{A}_k$ and G a connected reductive linear algebraic group over k. Let $\pi = \bigotimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A}_k)$. We assume that G is quasi-split over k, π occurs with multiplicity one in the automorphic discrete spectrum of G, and π is globally generic, i.e. the Whittaker function W_f is non-zero for some $f \in \pi$. Then the Lapid–Mao conjecture [LM15] predicts that

$$\langle f, f\rangle = |\mathcal{S}_\pi| \cdot \frac{L^S(1, \pi, \mathrm{Ad})}{\Delta_G^S} \cdot \prod_{v \in S} \alpha_v(f_v)^{-1}$$

for any $f = \bigotimes_v f_v \in \pi$ normalized so that $W_f(1) = 1$. Here $\langle f, f\rangle$ is the Petersson norm of f, $\mathcal{S}_\pi$ is the global component group of the (conjectural) Arthur parameter of π, S is a sufficiently large finite set of places of k, $L^S(s, \pi, \mathrm{Ad})$ is the partial adjoint L-function of π, Δ_G^S is a special value of some partial L-function depending only on G, and $\alpha_v(f_v)$ is the Whittaker integral of the matrix coefficient associated to $f_v \in \pi_v$. In fact, Lapid and Mao deduced the conjecture for $G = \mathrm{GL}_n$ from the theory of Rankin-Selberg integrals developed by Jacquet, Piatetski-Shapiro, and Shalika. Moreover, in [LM17], they established an analogous formula when

G is the metaplectic group Mp_{2n} (which is a nonlinear two-fold cover of the symplectic group Sp_{2n}) under the assumption that π_v is a discrete series representation for all archimedean places v of k. Recently, Furusawa and Morimoto proved the conjecture for $G = \mathrm{GSp}_4$ in [FM22, Theorem 6.3]. However, to derive from the Lapid-Mao conjecture an explicit formula for $\langle f, f\rangle$ suitable for applications, it is necessary to carry out the computation of the local integral $\alpha_v(f_v)$, which is extremely hard when v is archimedean.

The purpose of this paper is to prove some explicit formula for Petersson norms when $G = \mathrm{PGSp}_4$. For simplicity, we assume in the introduction that $k = \mathbb{Q}$, π_p is unramified for all primes p, π_∞ is either a discrete series representation or a spherical principal series representation, and $f \in \pi$ is the normalized new vector (see below for the details). Although the Lapid-Mao conjecture has been proved in this case, it seems impossible to compute $\alpha_\infty(f_\infty)$ directly. Nevertheless, by adopting a different approach and using reduction to the endoscopic case, we establish an explicit formula for $\langle f, f\rangle$, which we now describe in more detail.

1.1. **An explicit formula for Petersson norms.** Let $\pi = \bigotimes_v \pi_v$ be an irreducible globally generic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with trivial central character. By [GT11, Theorem 12.1] (see also [CKPSS04] and [AS06]), π has a strong functorial lift Π to $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$. We say that π is stable (resp. endoscopic) if Π is cuspidal (resp. non-cuspidal). We assume π_p is unramified for all primes p, and consider the following two cases:

(DS)

$$\pi_\infty|_{\mathrm{Sp}_4(\mathbb{R})} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)},$$

where $D_{(\lambda_1,\lambda_2)}$ is the (limit of) discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ with Blattner parameter $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ such that $1 - \lambda_1 \le \lambda_2 \le 0$.

(PS)

$$\pi_\infty|_{\mathrm{Sp}_4(\mathbb{R})} = \mathrm{Ind}^{\mathrm{Sp}_4(\mathbb{R})}_{\mathrm{Sp}_4(\mathbb{R})\cap\mathbf{B}(\mathbb{R})}(|\ |^{\lambda_1} \boxtimes |\ |^{\lambda_2})$$

for some $\lambda_1, \lambda_2 \in \mathbb{C}$.

Here $\mathbf{B}$ is the standard Borel subgroup of GSp_4.

Let $f = \bigotimes_v f_v \in \pi$ be a non-zero cusp form satisfying the following conditions:

(1.1)
- f_p is $\mathrm{GSp}_4(\mathbb{Z}_p)$-invariant for all primes p.
- In Case (DS), f_∞ is a lowest weight vector of the minimal $\mathrm{U}(2)$-type of $D_{(-\lambda_2,-\lambda_1)}$.
- In Case (PS), f_∞ is an $(\mathrm{Sp}_4(\mathbb{R}) \cap \mathrm{O}(4))$-invariant vector.

For the choice of f_∞ in Case (DS), we refer to § 2.4 for more detail. By [JS07], the multiplicity of π in the space of cusp forms on $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ is one. Hence the conditions in (1.1) uniquely determine $f \in \pi$ up to scalars. Let U be the unipotent radical of $\mathbf{B}$ and ψ_U the standard non-degenerate character of $U(\mathbb{Q})\backslash U(\mathbb{A}_\mathbb{Q})$ (see § 2.1 for precise definition). The Whittaker function of f with respect to ψ_U is defined by

$$W(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A}_\mathbb{Q})} f(ug)\overline{\psi_U(u)}\, du.$$

Here du is the Tamagawa measure on $U(\mathbb{A}_\mathbb{Q})$. We may decompose W into a product of local Whittaker functions $W = \prod_v W_v$. By explicit formulae for Whittaker functions [CS80], [Oda94], [Mor04], [Ish05], and [RS07], $W_v(1) \neq 0$ for all places v. We normalize f as follows (see also Remark 1.3 below): Let $W_p(1) = 1$ for all primes p. In Case (DS), let

$$W_\infty(1) = e^{-2\pi} \int_{c_1-\sqrt{-1}\infty}^{c_1+\sqrt{-1}\infty} \frac{ds_1}{2\pi\sqrt{-1}} \int_{c_2-\sqrt{-1}\infty}^{c_2+\sqrt{-1}\infty} \frac{ds_2}{2\pi\sqrt{-1}} (4\pi^3)^{(-s_1+\lambda_1+1)/2}(4\pi)^{(-s_2+\lambda_2)/2} \times \Gamma\left(\frac{s_1+s_2-2\lambda_2+1}{2}\right)\Gamma\left(\frac{s_1+s_2+1}{2}\right)\Gamma\left(\frac{s_1}{2}\right)\Gamma\left(\frac{-s_2}{2}\right), \tag{1.2}$$

where $c_1, c_2 \in \mathbb{R}$ satisfy $c_1 + c_2 + 1 > 0$ and $c_1 > 0 > c_2$. In Case (PS), let

$$\begin{aligned}
W_\infty(1) = & \int_{c_1-\sqrt{-1}\infty}^{c_1+\sqrt{-1}\infty} \frac{ds_1}{2\pi\sqrt{-1}} \int_{c_2-\sqrt{-1}\infty}^{c_2+\sqrt{-1}\infty} \frac{ds_2}{2\pi\sqrt{-1}} 2^{-4}\pi^{-s_1-s_2} \\
& \times \Gamma\left(\frac{s_1+\lambda_1}{2}\right)\Gamma\left(\frac{s_1-\lambda_1}{2}\right)\Gamma\left(\frac{s_1+\lambda_2}{2}\right)\Gamma\left(\frac{s_1-\lambda_2}{2}\right) \\
& \times \Gamma\left(\frac{s_2}{2}+\frac{\lambda_1+\lambda_2}{4}\right)\Gamma\left(\frac{s_2}{2}-\frac{\lambda_1+\lambda_2}{4}\right)\Gamma\left(\frac{s_2}{2}+\frac{\lambda_1-\lambda_2}{4}\right)\Gamma\left(\frac{s_2}{2}-\frac{\lambda_1-\lambda_2}{4}\right) \\
& \times \Gamma\left(\frac{s_1+s_2}{2}+\frac{\lambda_1+\lambda_2}{4}\right)^{-1}\Gamma\left(\frac{s_1+s_2}{2}-\frac{\lambda_1+\lambda_2}{4}\right)^{-1} \\
& \times {}_3F_2\left(\frac{s_1}{2}, \frac{s_2}{2}+\frac{\lambda_1-\lambda_2}{4}, \frac{s_2}{2}-\frac{\lambda_1-\lambda_2}{4}; \frac{s_1+s_2}{2}+\frac{\lambda_1+\lambda_2}{4}, \frac{s_1+s_2}{2}-\frac{\lambda_1+\lambda_2}{4}; 1\right),
\end{aligned} \tag{1.3}$$

where ${}_3F_2$ is a generalized hypergeometric function and $c_1, c_2 \in \mathbb{R}$ satisfy

$$c_1 > \max\{|\mathrm{Re}(\lambda_1)|, |\mathrm{Re}(\lambda_2)|\}, \quad c_2 > \max\left\{\left|\mathrm{Re}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|, \left|\mathrm{Re}\left(\frac{\lambda_1-\lambda_2}{2}\right)\right|\right\}.$$

Note that by [Ish05, Theorem 3.2 and Proposition 3.5], the right-hand side of (1.3) depends only on the orbit of (λ_1, λ_2) under the action of the Weyl group. Let $\langle f, f\rangle$ be the Petersson norm of f defined by

$$\langle f, f\rangle = \int_{\mathbb{A}_\mathbb{Q}^\times \mathrm{GSp}_4(\mathbb{Q})\backslash \mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})} |f(g)|^2\, dg.$$

Here dg is the Tamagawa measure on $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$.

Theorem 1.1. *We have*

$$\langle f, f\rangle = 2^c \cdot \frac{L(1, \pi, \mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot C_\infty.$$

Here Ad *is the adjoint representation of* $\mathrm{GSp}_4(\mathbb{C})$ *on* $\mathfrak{pgsp}_4(\mathbb{C})$,

$$\begin{aligned}
\Delta_{\mathrm{PGSp}_4} &= \xi(2)\xi(4), \\
c &= \begin{cases} 1 & \text{if } \pi \text{ is stable}, \\ 2 & \text{if } \pi \text{ is endoscopic}, \end{cases} \\
C_\infty &= \begin{cases} 2^{\lambda_1-\lambda_2+5}\pi^{3\lambda_1-\lambda_2+5}(1+\lambda_1-\lambda_2)^{-1} & \text{in Case (DS)}, \\ 2^{-4} & \text{in Case (PS)}, \end{cases}
\end{aligned}$$

where $\xi(s)$ *is the completed Riemann zeta function.*

Remark 1.2. In Case (DS), $1 + \lambda_1 - \lambda_2$ is the dimension of the minimal U(2)-type of $D_{(-\lambda_2, -\lambda_1)}$.

Remark 1.3. Recall that, in the case of $\mathrm{GL}_2(\mathbb{R})$, we may normalize the Whittaker function so that its value at $\mathrm{diag}(a, 1)$ with $a > 0$ is given by

$$a^{\kappa/2}e^{-2\pi a} = \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} (2\pi)^{-s-\kappa/2}\Gamma\left(s+\frac{\kappa}{2}\right) a^{-s}\, ds$$

for the (limit of) discrete series representation with minimal weight $\kappa \in \mathbb{Z}_{\geq 1}$ and

$$a^{1/2}K_\lambda(2\pi a) = \frac{1}{2\pi\sqrt{-1}} \int_{c-\sqrt{-1}\infty}^{c+\sqrt{-1}\infty} 2^{-2}\pi^{-s-1/2}\Gamma\left(\frac{s+\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s-\lambda}{2}+\frac{1}{4}\right) a^{-s}\, ds$$

for the spherical principal series representation $\mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^\lambda \boxtimes |\ |^{-\lambda})$, where B is the standard Borel subgroup of GL_2. Then the corresponding Jacquet-Langlands' local zeta integral for GL_2 gives the standard L-factor. Similarly, if we normalize the Whittaker function on $\mathrm{GSp}_4(\mathbb{R})$ as in (1.2) and (1.3), then the corresponding Novodvorsky's local zeta integral for GSp_4 gives the spinor L-factor (cf. [Mor04] and [IM08]). It would also be interesting to compare our normalization in Case (DS) with the rational structures studied by Harris and Kudla [HK92].

Remark 1.4. In Case (DS), Theorem 1.1 plays an important role in the result of Lemma and Ochiai [LO18] on the congruence between a fixed irreducible globally generic endoscopic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ and irreducible stable cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$.

1.2. **An outline of the proof.** The proof of Theorem 1.1 consists of two steps.

- We first prove that there exists a constant $\widetilde{C}_\infty$ depending only on π_∞ such that

$$\langle f, f\rangle = 2^c \cdot \frac{L(1,\pi,\mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot \widetilde{C}_\infty. \tag{1.4}$$

For this, we consider the automorphic L-function

$$L(s,\pi\times\pi^\vee) = \xi(s)\cdot L(s,\pi,\mathrm{std})\cdot L(s,\pi,\mathrm{Ad}),$$

where $\pi^\vee$ is the contragredient representation of π and $L(s,\pi,\mathrm{std})$ is the standard L-function of π. Recall the integral representations (ignoring factors which do not depend on π)

$$\langle \mathcal{E}(s), \bar{f}\otimes f\rangle = L\left(\frac{s+1}{2}, \pi\times\pi^\vee\right)\cdot \mathcal{Z}(s,\pi_\infty),$$
$$\langle E(s), \bar{f}\otimes f\rangle = \langle f, f\rangle \cdot L\left(s+\frac{1}{2}, \pi, \mathrm{std}\right)\cdot Z(s,\pi_\infty)$$

due to Jiang [Jia96] and Piatetski-Shapiro and Rallis [PSR87], respectively. Here $\mathcal{E}(s)$ and $E(s)$ are Eisenstein series on GSp_8 induced from characters of Levi subgroups $\mathrm{GL}_3\times\mathrm{GSp}_2$ and $\mathrm{GL}_4\times\mathrm{GL}_1$, respectively, and $\mathcal{Z}(s,\pi_\infty)$ and $Z(s,\pi_\infty)$ are the associated archimedean zeta integrals. Recall also that these Eisenstein series have Laurent expansions of the form

$$\mathcal{E}(s) = \frac{\mathcal{E}_{-2}(1)}{(s-1)^2} + \frac{\mathcal{E}_{-1}(1)}{s-1} + \cdots,$$
$$E(s) = \frac{E_{-1}(\frac{1}{2})}{s-\frac{1}{2}} + E_0\left(\frac{1}{2}\right) + \cdots.$$

By the Siegel-Weil formula [GQT14], we have

$$\mathcal{E}_{-2}(1) = E_{-1}\left(\frac{1}{2}\right),$$
$$\mathcal{E}_{-1}(1) = E_0\left(\frac{1}{2}\right) + \mathbb{E}_0\left(\frac{1}{2}\right) + \widetilde{E}_{-1}\left(\frac{1}{2}\right)$$

for some Eisenstein series $\mathbb{E}(s)$ and $\widetilde{E}(s)$ on GSp_8 induced from characters of Levi subgroups $\mathrm{GL}_2\times\mathrm{GSp}_4$ and $\mathrm{GL}_4\times\mathrm{GL}_1$, respectively. Then we have

$$\langle \mathcal{E}_{-2}(1), \bar{f}\otimes f\rangle = \left\langle E_{-1}\left(\frac{1}{2}\right), \bar{f}\otimes f\right\rangle.$$

In fact, this identity says $0=0$ when π is stable. Moreover, when π is stable, we show that

$$\left\langle \mathbb{E}_0\left(\frac{1}{2}\right), \bar{f}\otimes f\right\rangle = \left\langle \widetilde{E}_{-1}\left(\frac{1}{2}\right), \bar{f}\otimes f\right\rangle = 0,$$

so that

$$\langle \mathcal{E}_{-1}(1), \bar{f}\otimes f\rangle = \left\langle E_0\left(\frac{1}{2}\right), \bar{f}\otimes f\right\rangle.$$

Combining this with the integral representations, and noting that $L(s,\pi,\mathrm{std})$ is holomorphic and non-zero (resp. has a simple pole) at $s=1$ if π is stable (resp. endoscopic), we can deduce (1.4) with

$$\widetilde{C}_\infty = \frac{1}{L(1,\pi_\infty,\mathrm{Ad})}\cdot\frac{\mathcal{Z}(1,\pi_\infty)}{Z(\frac{1}{2},\pi_\infty)}.$$

- We next show that

$$\widetilde{C}_\infty = C_\infty. \tag{1.5}$$

This is a local problem, but we address it by a global argument. Suppose that π is endoscopic. Recall that π is of type (DS) or (PS) at the archimedean place and unramified at all finite places. Since π is endoscopic, it can be realized as a global theta lift from the orthogonal similitude group $\mathrm{GO}_{2,2}$. Hence, by using the Rallis inner product formula [GQT14], we can show that

$$\langle f, f \rangle = 2^c \cdot \frac{L(1,\pi,\mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot C'_\infty \tag{1.6}$$

for some constant C'_∞ defined by doubling local zeta integral depending only on the archimedean component. Moreover, by explicit computation, we have

$$C'_\infty = C_\infty. \tag{1.7}$$

Therefore, if π_∞ is the archimedean component of such π, then we can deduce (1.5) for π_∞ from (1.4), (1.6), and (1.7). Now we take an arbitrary representation π_∞ of $\mathrm{GSp}_4(\mathbb{R})$ of type (DS) or (PS). When π_∞ is of type (DS) and sufficiently regular, the existence of its globalization π is clear. When π_∞ is of type (PS), we can deduce from the limit multiplicity formula [Shi12] that there exists π whose archimedean component is arbitrarily close to π_∞. (Strictly speaking, we need to consider cusp forms of square-free level over totally real number fields.) Thus we may conclude (1.5) for π_∞ by showing that $\widetilde{C}_\infty = \widetilde{C}_\infty(\pi_\infty)$ is analytic as a function of π_∞. Finally, when π_∞ is of type (DS) without regularity assumption, we consider cusp forms of square-free level in the global argument. In this case, the formulas analogous to (1.4) and (1.6) still hold but some extra constants depending only on the non-archimedean components appear in the formulas. Thus we also need to determine these constants, but by a similar argument using the limit multiplicity formula [Shi12], we are reduced to compute certain non-archimedean doubling local zeta integral.

Acknowledgement. The authors would like to thank Valentin Blomer, Yao Cheng, Masaaki Furusawa, Ming-Lun Hsieh, Tamotsu Ikeda, Francesco Lemma, and Tadashi Ochiai for their advice and encouragement. The first-named author is partially supported by MOST GSSAP with Grant Number 2917-I-002-003. The second-named author is partially supported by JSPS KAKENHI Grant Number 26287003.

2. Main result

2.1. **Notation.** Fix a totally real number field k. Let $\mathfrak{o}$, $\mathfrak{d}$, and $\mathfrak{D}$ be the ring of integers, the different ideal, and the absolute discriminant of k, respectively. Let $\mathbb{A} = \mathbb{A}_k$ be the ring of adeles of k and $\mathbb{A}_f$ its finite part. For a finite dimensional vector space V over k, let $\mathcal{S}(V(\mathbb{A}))$ be the space of Schwartz functions on $V(\mathbb{A})$. Let $\psi_0 = \bigotimes_v \psi_{v,0}$ be the standard additive character of $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ defined so that

$$\psi_{p,0}(x) = e^{-2\pi\sqrt{-1}x} \text{ for } x \in \mathbb{Z}[p^{-1}],$$
$$\psi_{\infty,0}(x) = e^{2\pi\sqrt{-1}x} \text{ for } x \in \mathbb{R}.$$

Let $\psi = \psi_k = \psi_0 \circ \mathrm{tr}_{k/\mathbb{Q}}$ be the standard additive character of $k\backslash\mathbb{A}$.

Let v be a place of k. We denote by ψ_v the v-component of ψ. If v is a finite place, let $\mathfrak{o}_v$, ϖ_v, and q_v be the maximal compact subring of k_v, a generator of the maximal ideal of $\mathfrak{o}_v$, and the cardinality of $\mathfrak{o}_v/\varpi_v\mathfrak{o}_v$. Let $|\ |_v$ be the absolute value on k_v normalized so that $|\varpi_v|_v = q_v^{-1}$. Let $\mathfrak{c}_v$ be the integer such that $\mathfrak{d}_v = \varpi_v^{\mathfrak{c}_v}\mathfrak{o}_v$. Note that $\mathfrak{c}_v$ is the largest integer so that ψ_v is trivial on $\varpi_v^{-\mathfrak{c}_v}\mathfrak{o}_v$. If v is a real place, let $|\ |_v$ be the ordinary absolute value on k_v.

Let $\xi(s) = \xi_k(s) = \prod_v \zeta_v(s)$ be the completed Dedekind zeta function of k, where v ranges over the places of k and

$$\zeta_v(s) = \begin{cases} (1-q_v^{-s})^{-1} & \text{if } v \text{ is finite,} \\ \pi^{-s/2}\Gamma(s/2) & \text{if } v \text{ is real.} \end{cases}$$

Here $\Gamma(s)$ is the gamma function. Define $\rho = \mathrm{Res}_{s=1}\xi(s)$. For a finite set S of places of k, let $\xi^S(s) = \prod_{v\notin S}\zeta_v(s)$ be the partial Dedekind zeta function of k.

If S is a set, then we let $\mathbb{I}_S$ be the characteristic function of S. Let $\mathrm{M}_{n,m}$ be the matrix algebra of n by m matrices. Let GSp_{2n} denote the symplectic similitude group of rank $n+1$ defined by

$$\mathrm{GSp}_{2n} = \left\{ g \in \mathrm{GL}_{2n} \;\middle|\; g \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} {}^t g = \nu(g) \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix},\ \nu(g) \in \mathbb{G}_m \right\}.$$

Let $\mathrm{Sp}_{2n} = \ker(\nu)$. Throughout this article, we write

$$G = \mathrm{GSp}_4$$

unless otherwise specified. Let Z_G be the center of G. Let

$$\mathbf{B} = \left\{ \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & \nu t_1^{-1} & 0 \\ 0 & 0 & * & \nu t_2^{-1} \end{pmatrix} \in G \;\middle|\; t_1, t_2, \nu \in \mathbb{G}_m \right\}$$

be the standard Borel subgroup of G, and let U be its unipotent radical. Let ψ_U be the non-degenerate character of $U(k)\backslash U(\mathbb{A})$ defined by

$$\psi_U\left(\begin{pmatrix} 1 & x & * & * \\ 0 & 1 & * & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \right) = \psi(-x-y).$$

Let $\mathbf{T} \subset \mathbf{B}$ be the standard maximal tours of G. In GL_2, let B be the Borel subgroup consisting of upper triangular matrices, and put

$$\mathbf{a}(\nu) = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{d}(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}, \quad \mathbf{m}(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad \mathbf{n}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for $\nu, t \in \mathbb{G}_m$ and $x \in \mathbb{G}_a$. Let

$$\mathrm{SO}(2) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

Let v be a finite place of k. Let $K_0(\varpi_v)$ be the Iwahori subgroup of $\mathrm{GL}_2(k_v)$ defined by

$$K_0(\varpi_v) = \begin{pmatrix} \mathfrak{o}_v & \mathfrak{o}_v \\ \varpi_v \mathfrak{o}_v & \mathfrak{o}_v \end{pmatrix} \cap \mathrm{GL}_2(\mathfrak{o}_v).$$

The paramodular group $\mathrm{K}(\varpi_v)$ of level $\varpi_v \mathfrak{o}_v$ is the subgroup of $G(k_v)$ consisting of $g \in G(k_v)$ such that $\nu(g) \in \mathfrak{o}_v^\times$ and

$$g \in \begin{pmatrix} \mathfrak{o}_v & \mathfrak{o}_v & \varpi_v^{-1}\mathfrak{o}_v & \mathfrak{o}_v \\ \varpi_v \mathfrak{o}_v & \mathfrak{o}_v & \mathfrak{o}_v & \mathfrak{o}_v \\ \varpi_v \mathfrak{o}_v & \varpi_v \mathfrak{o}_v & \mathfrak{o}_v & \varpi_v \mathfrak{o}_v \\ \varpi_v \mathfrak{o}_v & \mathfrak{o}_v & \mathfrak{o}_v & \mathfrak{o}_v \end{pmatrix}.$$

For $\nu \in \mathbb{C}$, let $K_\nu(z)$ be the modified Bessel function defined by

$$K_\nu(z) = \frac{1}{2} \int_0^\infty e^{-z(t+t^{-1})/2} t^{\nu-1}\, dt \tag{2.1}$$

if $\mathrm{Re}(z) > 0$.

2.2. **Measures.** Let v be a place of k. If v is finite, we normalize the Haar measures on k_v and $k_v^\times$ so that $\mathrm{vol}(\mathfrak{o}_v) = 1$ and $\mathrm{vol}(\mathfrak{o}_v^\times) = 1$, respectively. If v is real, we normalize the Haar measures on $k_v \simeq \mathbb{R}$ and $k_v^\times \simeq \mathbb{R}^\times$ so that $\mathrm{vol}([1,2]) = 1$ and $\mathrm{vol}([1,2]) = \log 2$, respectively. Let m be a positive integer. Let dg_v be the Haar measure on $\mathrm{GL}_m(k_v)$ defined as follows: For $\phi \in L^1(\mathrm{GL}_m(k_v))$, we have

(2.2)

$$\int_{\mathrm{GL}_m(k_v)} \phi(g_v)\, dg_v = \prod_{1 \le i < j \le m} \int_{k_v} du_{ij} \prod_{1 \le i \le m} \int_{k_v^\times} d^\times t_i \int_{K_v} dk\, \phi\left(\begin{pmatrix} t_1 & u_{12} & \cdots & u_{1m} \\ 0 & t_2 & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_m \end{pmatrix} k \right) \prod_{1 \le i \le m} |t_i|_v^{-m+i},$$

where

$$K_v = \begin{cases} \mathrm{GL}_m(\mathfrak{o}_v) & \text{if } v \text{ is finite,} \\ \mathrm{O}(m) & \text{if } v \text{ is real,} \end{cases}$$

and $\mathrm{vol}(K_v) = 1$. Let H be a connected reductive linear algebraic group defined and split over k_v. Fix a Chevalley basis of $\mathrm{Lie}(H)$. The basis determines a top differential form on H over $\mathbb{Z}$ which is unique up to ± 1. The top differential form together with the self-dual Haar measure on k_v with respect to ψ_v determines a Haar measure on $H(k_v)$ as explained in [Vos96, §6]. We call it the local Tamagawa measure on $H(k_v)$.

For a connected linear algebraic group H over k, we take the Tamagawa measure on $H(\mathbb{A})$ (cf. [Vos96, §6]). For any compact group K, we take the Haar measure on K such that $\mathrm{vol}(K) = 1$. Let dg be the Tamagawa measure on $\mathrm{GL}_m(\mathbb{A})$. Then (cf. [Lai80, §5])

$$dg = \mathfrak{D}^{-m^2/2} \rho^{-1} \prod_{i=2}^{m} \xi(i)^{-1} \cdot \prod_v dg_v. \tag{2.3}$$

2.3. **Automorphic representations of** GSp_4**.** Let $\pi = \bigotimes_v \pi_v$ be an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. By [CKPSS04], [AS06], [GT11, Theorem 12.1], π has a strong functorial lift Π to $\mathrm{GL}_4(\mathbb{A})$. By [GRS01], [GT11, Theorem 12.1], either Π is cuspidal or $\Pi = \sigma_1 \boxplus \sigma_2$ for some irreducible cuspidal automorphic representations σ_1 and σ_2 of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character such that $\sigma_1 \neq \sigma_2$. We say that π is stable (resp. endoscopic) if Π is cuspidal (resp. non-cuspidal).

Recall that the dual group of G is $\mathrm{GSp}_4(\mathbb{C})$. Let Ad denote the adjoint representation of $\mathrm{GSp}_4(\mathbb{C})$ on $\mathfrak{pgsp}_4(\mathbb{C})$, and std the composition of the projection $\mathrm{GSp}_4(\mathbb{C}) \to \mathrm{PGSp}_4(\mathbb{C})$ with the standard representation of $\mathrm{PGSp}_4(\mathbb{C}) \simeq \mathrm{SO}_5(\mathbb{C})$ on $\mathbb{C}^5$. Let S be a finite set of places of k including the archimedean places such that, for $v \notin S$, π_v is unramified. Then the partial adjoint and standard L-functions of π are defined as the Euler products

$$L^S(s, \pi, \mathrm{Ad}) = \prod_{v \notin S} L(s, \pi_v, \mathrm{Ad}), \quad L^S(s, \pi, \mathrm{std}) = \prod_{v \notin S} L(s, \pi_v, \mathrm{std})$$

for $s \in \mathbb{C}$, which are absolutely convergent for $\mathrm{Re}(s)$ sufficiently large. Also, we have

$$L^S(s, \Pi, \mathrm{Sym}^2) = L^S(s, \pi, \mathrm{Ad}),$$
$$L^S(s, \Pi, \wedge^2) = \xi^S(s) L^S(s, \pi, \mathrm{std}).$$

In particular, $L^S(s, \pi, \mathrm{Ad})$ and $L^S(s, \pi, \mathrm{std})$ admit meromorphic continuations to $\mathbb{C}$. (In a more general context, the meromorphic continuation of $L^S(s, \pi, \mathrm{std})$ was established by Piatetski-Shapiro and Rallis [PSR87] much earlier.) By [GRS01], [GT11, Theorem 12.1], $L^S(s, \Pi, \wedge^2)$ has a simple (resp. double) pole at $s = 1$ if π is stable (resp. endoscopic). Hence $L^S(s, \pi, \mathrm{std})$ is holomorphic and non-zero (resp. has a simple pole) at $s = 1$ if π is stable (resp. endoscopic). Moreover, $L^S(s, \pi, \mathrm{Ad})$ is holomorphic and non-zero at $s = 1$.

For any place v of k, we denote by $\Phi_v : L_{k_v} \to \mathrm{GSp}_4(\mathbb{C})$ the local L-parameter attached to π_v by the local Langlands correspondence established by Gan and Takeda [GT11] if v is finite and by Langlands [Lan89] if v is real. Here L_{k_v} is the Weil–Deligne group of k_v if v is finite but the Weil group of k_v if v is real. Since Π_v is unitary and generic (and hence "almost tempered"), the adjoint L-factor

$$L(s, \pi_v, \mathrm{Ad}) = L(s, \mathrm{Ad} \circ \Phi_v)$$

defined as in [Tat79, §3] is holomorphic at $s = 1$. In fact, the same holds for any irreducible generic admissible representation of $G(k_v)$ (see [GP92, Conjecture 2.6], [AS08], [GT11], [GI16, Proposition B.1]). Hence the completed adjoint L-function $L(s, \pi, \mathrm{Ad})$ is holomorphic and non-zero at $s = 1$.

2.4. **Main result.** Let $\pi = \bigotimes_v \pi_v$ be an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Denote by $\mathfrak{n} \trianglelefteq \mathfrak{o}$ the paramodular conductor of π (cf. [RS07]). We assume π satisfies the following conditions:

- $\mathfrak{n}$ is square-free.
- For each real place v, π_v is in one of the following two types:

(DS)

$$\pi_v|_{\mathrm{Sp}_4(k_v)} = D_{(\lambda_{1,v}, \lambda_{2,v})} \oplus D_{(-\lambda_{2,v}, -\lambda_{1,v})},$$

where $D_{(\lambda_{1,v}, \lambda_{2,v})}$ is the (limit of) discrete series representation of $\mathrm{Sp}_4(k_v)$ with Blattner parameter $(\lambda_{1,v}, \lambda_{2,v}) \in \mathbb{Z}^2$ such that $1 - \lambda_{1,v} \leq \lambda_{2,v} \leq 0$.

(PS)

$$\pi_v|_{\mathrm{Sp}_4(k_v)} = \mathrm{Ind}_{\mathrm{Sp}_4(k_v)\cap \mathbf{B}(k_v)}^{\mathrm{Sp}_4(k_v)}(|\ |_v^{\lambda_{1,v}} \boxtimes |\ |_v^{\lambda_{2,v}})$$

for some $\lambda_{1,v}, \lambda_{2,v} \in \mathbb{C}$.

In Case (DS), we follow [Mor04] for the choice of the Cartan subalgebra in $\mathfrak{sp}_4(\mathbb{R})$ and the positive systems.

Denote by $S(\mathfrak{n})$ the set of finite places dividing $\mathfrak{n}$, by $S(\mathrm{DS})$ and $S(\mathrm{PS})$ the sets of real places of type (DS) and (PS), respectively.

Let $f = \bigotimes_v f_v \in \pi$ be a non-zero cusp form satisfying the following conditions:

$$\begin{aligned} &f_v \text{ is } G(\mathfrak{o}_v)\text{-invariant for all finite places } v \notin S(\mathfrak{n}). \\ &f_v \text{ is } \mathrm{K}(\varpi_v)\text{-invariant for all } v \in S(\mathfrak{n}). \\ &f_v \text{ is a lowest weight vector of the minimal } \mathrm{U}(2)\text{-type of } D_{(-\lambda_{2,v},-\lambda_{1,v})} \text{ for all } v \in S(\mathrm{DS}). \\ &f_v \text{ is a } (\mathrm{Sp}_4(k_v)\cap \mathrm{O}(4))\text{-invariant vector for all } v \in S(\mathrm{PS}). \end{aligned} \tag{2.4}$$

For $v \in S(\mathrm{DS})$, the vector f_v is proportional to the vector v_0 in the notation of [Mor04, §1.2]. By [JS07], the multiplicity of π in the space of cusp forms on $G(\mathbb{A})$ is one. Therefore, the conditions (2.4) characterize $f \in \pi$ up to scalars. Let W be the Whittaker function of f with respect to ψ_U defined by

$$W(g) = \int_{U(k)\backslash U(\mathbb{A})} f(ug)\overline{\psi_U(u)}\, du.$$

Here du is the Tamagawa measure on $U(\mathbb{A})$. We may decompose $W = \prod_v W_v$ as a product of local Whittaker functions of π_v with respect to $\psi_{U,v}$. We normalize f as follows:

$$\begin{aligned} &W_v(\mathrm{diag}(\varpi_v^{-\mathfrak{c}_v}, 1, \varpi_v^{2\mathfrak{c}_v}, \varpi_v^{\mathfrak{c}_v})) = 1 \text{ for all finite places } v, \\ &W_v(1) \text{ is normalized as in (1.2) for all } v \in S(\mathrm{DS}), \\ &W_v(1) \text{ is normalized as in (1.3) for all } v \in S(\mathrm{PS}). \end{aligned} \tag{2.5}$$

Let $\langle f, f\rangle$ be the Petersson norm of f defined by

$$\langle f, f\rangle = \int_{Z_G(\mathbb{A})G(k)\backslash G(\mathbb{A})} |f(g)|^2\, dg.$$

Our main result is the following theorem.

Theorem 2.1. *We have*

$$\langle f, f\rangle = 2^c \cdot \frac{L(1,\pi,\mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot \prod_v C(\pi_v). \tag{2.6}$$

Here

$$\Delta_{\mathrm{PGSp}_4} = \xi(2)\xi(4),$$

$$c = \begin{cases} 1 & \text{if } \pi \text{ is stable}, \\ 2 & \text{if } \pi \text{ is endoscopic}, \end{cases}$$

$$C(\pi_v) = \begin{cases} q_v^{-5\mathfrak{c}_v} & \text{if } v \nmid \infty\mathfrak{n}, \\ q_v^{-1-5\mathfrak{c}_v}\zeta_v(2)^{-1}\zeta_v(4) & \text{if } v \in S(\mathfrak{n}), \\ 2^{\lambda_{1,v}-\lambda_{2,v}+5}\pi^{3\lambda_{1,v}-\lambda_{2,v}+5}(1+\lambda_{1,v}-\lambda_{2,v})^{-1} & \text{if } v \in S(\mathrm{DS}), \\ 2^{-4} & \text{if } v \in S(\mathrm{PS}). \end{cases}$$

Remark 2.2. For any irreducible globally generic cuspidal automorphic representation π of $G(\mathbb{A})$, we have a formula for the Petersson norm as in (2.6) (cf. Proposition 5.4 below). But our assumption on π will be used to determine the constant $C(\pi_v)$ explicitly.

3. Siegel-Weil formula

In this section, we introduce Eisenstein series on $\mathrm{Sp}_{2n}(\mathbb{A})$, theta integrals and its regularization, and recall the Siegel-Weil formula in the second term range for $n = 4$. The results and notation in this section will be used in § 4.

3.1. **Eisenstein series.** Let n, r be positive integers such that $n \geq r$. Let

$$P_{n,r} = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & \nu(g)\,{}^t a^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix} \in \mathrm{GSp}_{2n} \;\middle|\; a \in \mathrm{GL}_r,\, g = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GSp}_{2n-2r} \right\}$$

be a maximal parabolic subgroup of GSp_{2n}. We define a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v$ of $\mathrm{GSp}_{2n}(\mathbb{A})$ by

$$\mathbf{K}_v = \begin{cases} \mathrm{GSp}_{2n}(\mathfrak{o}_v) & \text{if } v \text{ is finite,} \\ \mathrm{GSp}_{2n}(k_v) \cap \mathrm{O}(2n) & \text{if } v \text{ is real.} \end{cases}$$

Let $K_v = \mathbf{K}_v \cap \mathrm{Sp}_{2n}(k_v)$ for each place v of k, and $K = \prod_v K_v$. Let $\mathfrak{g}_\infty$ be the complexified Lie algebra of $\prod_{v|\infty} \mathrm{Sp}_{2n}(k_v)$ and $K_\infty = \prod_{v|\infty} K_v$ a maximal compact subgroup of $\prod_{v|\infty} \mathrm{Sp}_{2n}(k_v)$.

For $s \in \mathbb{C}$, let $I_{n,r}(s) = \mathrm{Ind}_{P_{n,r}(\mathbb{A})}^{\mathrm{GSp}_{2n}(\mathbb{A})}(\delta_{P_{n,r}}^{s/(2n-r+1)})$ denote the degenerate principal series representation of $\mathrm{GSp}_{2n}(\mathbb{A})$. Here $\delta_{P_{n,r}}$ is the modulus character of $P_{n,r}(\mathbb{A})$. For a holomorphic section F of $I_{n,r}(s)$, define an Eisenstein series $E^{(n,r)}(s,F)$ by

$$E^{(n,r)}(g; s, F) = \sum_{\gamma \in P_{n,r}(k) \backslash \mathrm{GSp}_{2n}(k)} F(\gamma g, s)$$

for $\mathrm{Re}(s) \gg 0$, and by the meromorphic continuation otherwise. Let

$$E^{(n,r)}(s,F) = \sum_{d \gg -\infty} (s - s_0)^d E_d^{(n,r)}(s_0, F)$$

be the Laurent expansion of $E^{(n,r)}(s,F)$ at $s = s_0$.

3.2. **Theta integrals.** In this section, we review the result of Kudla and Rallis [KR94] on the regularization of theta integrals.

Assume that $r \geq 2$. Let $V_{r,r} = k^{2r}$ be the space of column vectors equipped with a non-degenerate symmetric bilinear form $(\ ,\ )$ given by

$$(x,y) = {}^t x \begin{pmatrix} 0 & \mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix} y$$

for $x, y \in V_{r,r}$. Let $G' = \mathrm{GO}_{r,r}$ denote the orthogonal similitude group of $V_{r,r}$ defined by

$$\mathrm{GO}_{r,r} = \left\{ g' \in \mathrm{GL}_{2r} \;\middle|\; {}^t g' \begin{pmatrix} 0 & \mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix} g' = \nu(g') \begin{pmatrix} 0 & \mathbf{1}_r \\ \mathbf{1}_r & 0 \end{pmatrix},\, \nu(g') \in \mathbb{G}_m \right\}.$$

Let $G'_1 = \mathrm{O}_{r,r}$. We define a maximal compact subgroup $\mathbf{K}' = \prod_v \mathbf{K}'_v$ of $G'(\mathbb{A})$ by

$$\mathbf{K}'_v = \begin{cases} \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & \varpi_v^{\mathfrak{c}_v} \mathbf{1}_r \end{pmatrix} G'(\mathfrak{o}_v) \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & \varpi_v^{-\mathfrak{c}_v} \mathbf{1}_r \end{pmatrix} & \text{if } v \text{ is finite,} \\ G'(k_v) \cap \mathrm{O}(2r) & \text{if } v \text{ is real.} \end{cases}$$

Let $K'_v = \mathbf{K}'_v \cap G'_1(k_v)$ for each place v of k, and $K' = \prod_v K'_v$. Let $r' \leq r$ be a non-negative integer. Let

$$P'_{r'} = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & a' & * & b' \\ 0 & 0 & \nu(g)\,{}^t a^{-1} & 0 \\ 0 & c' & * & d' \end{pmatrix} \in G' \;\middle|\; a \in \mathrm{GL}_{r-r'},\, g = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{GO}_{r',r'} \right\}$$

be a maximal parabolic subgroup of G'. We denote by $M'_{r'}$ and $N'_{r'}$ the standard Levi subgroup and unipotent radical of $P'_{r'}$, respectively. Let dg' be the Haar measure on $G'_1(\mathbb{A})$ such that $\mathrm{vol}(G'_1(k)\backslash G'_1(\mathbb{A})) = 1$, dm' (resp. dn') the Tamagawa measure on $M'_{r'}(\mathbb{A})$ (resp. $N'_{r'}(\mathbb{A})$), and dk' the Haar measure on K' such that $\mathrm{vol}(K') = 1$. Define a constant $\kappa_{r,r'}$ by

$$\int_{G'_1(\mathbb{A})} \phi(g')\, dg' = \kappa_{r,r'} \int_{M'_{r'}(\mathbb{A}) \times N'_{r'}(\mathbb{A}) \times K'} \phi(m'n'k')\, dm'\, dn'\, dk'$$

for $\phi \in L^1(G'_1(\mathbb{A}))$.

Assume $n \geq r$. Let $\omega = \omega_{\psi, V_{r,r}, n}$ denote the Weil representation of $\mathrm{Sp}_{2n}(\mathbb{A}) \times G_1'(\mathbb{A})$ on $\mathcal{S}(V_{r,r}^n(\mathbb{A})) = \mathcal{S}(\mathrm{M}_{2r,n}(\mathbb{A}))$ with respect to ψ. We recall the explicit formula for ω as follows:

- for $g' \in G_1'(\mathbb{A})$, we have
$$\omega(1, g')\varphi(x) = \varphi(g'^{-1}x);$$
- for $a \in \mathrm{GL}_n(\mathbb{A})$, we have
$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, 1\right)\varphi(x) = |\det(a)|_{\mathbb{A}}^r \varphi(xa);$$
- for $b \in \mathrm{M}_{n,n}(\mathbb{A})$ with $b = {}^t b$, we have
$$\omega\left(\begin{pmatrix} \mathbf{1}_n & b \\ 0 & \mathbf{1}_n \end{pmatrix}, 1\right)\varphi(x) = \psi(\mathrm{tr}(\tfrac{1}{2}b(x,x)))\varphi(x);$$
- we have
$$\omega\left(\begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}, 1\right)\varphi(x) = \int_{V_{r,r}^n(\mathbb{A})} \psi(-\mathrm{tr}(x,y))\varphi(y)\, dy,$$
where dy is the Tamagawa measure on $V_{r,r}^n(\mathbb{A})$.

Let
$$\mathrm{G}(\mathrm{Sp}_{2n} \times G_1') = \{(g, g') \in \mathrm{GSp}_{2n} \times G' \mid \nu(g) = \nu(g')\}.$$
We extend ω to a representation of $\mathrm{G}(\mathrm{Sp}_{2n} \times G_1')(\mathbb{A})$ as follows:

$$\omega(g, g')\varphi = \omega\left(g \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \nu(g')^{-1}\mathbf{1}_n \end{pmatrix}, 1\right) L(g')\varphi \tag{3.1}$$

for $(g, g') \in \mathrm{G}(\mathrm{Sp}_{2n} \times G_1')(\mathbb{A})$ and $\varphi \in \mathcal{S}(V_{r,r}^n(\mathbb{A}))$. Here
$$L(g')\varphi(x) = |\nu(g')|_{\mathbb{A}}^{-nr/2}\varphi(g'^{-1}x).$$
Let $S(V_{r,r}^n(\mathbb{A}))$ be the subspace of $\mathcal{S}(V_{r,r}^n(\mathbb{A}))$ consisting of functions which correspond to polynomials in the Fock model at the archimedean places. For $(g, g') \in \mathrm{G}(\mathrm{Sp}_{2n} \times G_1')(\mathbb{A})$ and $\varphi \in S(V_{r,r}^n(\mathbb{A}))$, let
$$\Theta(g, g'; \varphi) = \sum_{x \in V_{r,r}^n(k)} \omega(g, g')\varphi(x).$$
Fix a real place v of k and let $z = z_{r-1,n} \in \mathfrak{z}(\mathfrak{g}_v)$ be the regularizing differential operator as in [KR94, Corollary 5.1.2], where $\mathfrak{z}(\mathfrak{g}_v)$ is the center of the universal enveloping algebra of $\mathfrak{g}_v$. By [KR94, Proposition 5.3.1], the function $g' \mapsto \Theta(g, g'; z \cdot \varphi)$ on $G_1'(k)\backslash G_1'(\mathbb{A})$ is rapidly decreasing. Here $\mathfrak{z}(\mathfrak{g}_v)$ acts on $S(V_{r,r}^n(\mathbb{A}))$ via the differential of ω.

Put $s_0' = (r-1)/2$. Let F be the $\mathbf{K}'$-invariant holomorphic section of $\mathrm{Ind}_{P_0'(\mathbb{A})}^{G'(\mathbb{A})}(\delta_{P_0'}^{s/(r-1)})$ such that $F(1, s) = 1$. Here $\delta_{P_0'}$ is the modulus character of $P_0'(\mathbb{A})$. Define an auxiliary Eisenstein series $E(s)$ by
$$E(g'; s) = \sum_{\gamma' \in P_0'(k)\backslash G'(k)} F(\gamma' g', s)$$
for $\mathrm{Re}(s) \gg 0$, and by the meromorphic continuation otherwise. Note that $E(s)$ has a simple pole at $s = s_0'$ with constant residue $\kappa_{r,0}$. Following [KR94, §5.5], we consider the integral
$$I^{(n,r)}(g; s, \varphi) = \kappa_{r,0}^{-1} Q_{n,r}(s)^{-1} \int_{G_1'(k)\backslash G_1'(\mathbb{A})} \Theta(g, g'h; z \cdot \varphi) E(g'h; s)\, dg',$$
where $\nu(h) = \nu(g)$ and
$$Q_{n,r}(s) = \prod_{i=0}^{r-1}((s - s_0' + i)^2 - (n+1-r)^2).$$
Let
$$I^{(n,r)}(s, \varphi) = \sum_{d \gg -\infty} (s - s_0')^d I_d^{(n,r)}(\varphi)$$
be the Laurent expansion of $I^{(n,r)}(s, \varphi)$ at $s = s_0'$. Note that $I^{(n,r)}(s, \varphi)$ has at most a simple (resp. double) pole at $s = s_0'$ if $r \leq (n+1)/2$ (resp. $(n+1)/2 < r \leq n$).

Let $\hat{\omega}$ be the Weil representation of $\mathrm{G}(\mathrm{Sp}_{2n} \times G'_1)(\mathbb{A})$ on $\mathcal{S}(\mathrm{M}_{r,2n}(\mathbb{A}))$ defined via the partial Fourier transform

$$\mathcal{S}(\mathrm{M}_{2r,n}(\mathbb{A})) \longrightarrow \mathcal{S}(\mathrm{M}_{r,2n}(\mathbb{A})), \quad \varphi \longmapsto \hat{\varphi},$$

where

$$\hat{\varphi}(u,v) = \int_{\mathrm{M}_{r,n}(\mathbb{A})} \varphi \begin{pmatrix} x \\ u \end{pmatrix} \psi(\mathrm{tr}(v\,{}^t x))\, dx$$

for $u, v \in \mathrm{M}_{r,n}(\mathbb{A})$. Here dx is the Tamagawa measure on $\mathrm{M}_{r,n}(\mathbb{A})$. For $\mathrm{Re}(s) > s'_0 - n + r$, define a $(\mathfrak{g}_\infty, K_\infty) \times \mathrm{GSp}_{2n}(\mathbb{A}_f)$-intertwining map

$$\mathcal{F} : S(V^n_{r,r}(\mathbb{A})) \longrightarrow I_{n,r}(s)$$

by

$$\mathcal{F}(\varphi)(g,s) = \int_{\mathrm{GL}_r(\mathbb{A})} \int_{K'} \hat{\omega}(g, k'g')\hat{\varphi}(0_{r\times n}, {}^t a, 0_{r\times(n-r)}) F(k'g', s) |\det(a)|_{\mathbb{A}}^{s - s'_0 + n}\, dk'\, da,$$

where $\nu(g') = \nu(g)$. Then, by [KR94, §5.5] and [GI11, §7.4],

$$I^{(n,r)}(s, \varphi) = E^{(n,r)}(s, \mathcal{F}(\varphi)).$$

We define the spherical Schwartz function $\varphi^o = \bigotimes_v \varphi^o_v \in S(V^n_{r,r}(\mathbb{A}))$ as follows:

- If v is finite, then

$$\varphi^o_v \begin{pmatrix} x \\ y \end{pmatrix} = q_v^{-\mathfrak{c}_v n r/2} \cdot \mathbb{I}_{\mathrm{M}_{r,n}(\varpi_v^{-\mathfrak{c}_v} \mathfrak{o}_v)}(x) \mathbb{I}_{\mathrm{M}_{r,n}(\mathfrak{o}_v)}(y).$$

- If v is real, then

$$\varphi^o_v(x) = e^{-\pi\, \mathrm{tr}(x\,{}^t x)}.$$

Note that

$$\omega_v(k,k')\varphi^o_v = \varphi^o_v \tag{3.2}$$

for $(k,k') \in (\mathbf{K}_v \times \mathbf{K}'_v) \cap \mathrm{G}(\mathrm{Sp}_{2n} \times G'_1)(k_v)$. For $\mathrm{Re}(s) > s'_0 - n + r - 1$, define an intertwining map

$$\mathcal{F}_v : S(V^n_{r,r}(k_v)) \longrightarrow I_{n,r,v}(s)$$

by

$$\mathcal{F}_v(\varphi)(g,s) = \int_{\mathrm{GL}_r(k_v)} \int_{K'_v} \hat{\omega}_v(g, k'_v g')\hat{\varphi}(0_{r\times n}, {}^t a_v, 0_{r\times(n-r)}) F_v(k'_v g', s) |\det(a_v)|_v^{s - s'_0 + n}\, dk'_v\, da_v,$$

where $\nu(g') = \nu(g)$ and F_v is the $\mathbf{K}'_v$-invariant holomorphic section of $\mathrm{Ind}^{G'(k_v)}_{P'_0(k_v)}(\delta^{s/(r-1)}_{P'_0})$ such that $F_v(1,s) = 1$. The Haar measure da_v on $\mathrm{GL}_r(k_v)$ is normalized as in (2.2). By a direct calculation,

$$\mathcal{F}_v(\varphi^o_v)(1,s) = \prod_{j=0}^{r-1} \zeta_v(s - s'_0 + n - j). \tag{3.3}$$

By (2.3), for $\varphi = \bigotimes_v \varphi_v \in S(V^n_{r,r}(\mathbb{A}))$, we have

$$\mathcal{F}(\varphi) = \mathfrak{D}^{-r^2/2} \rho^{-1} \prod_{i=2}^{r} \xi(i)^{-1} \cdot \bigotimes_v \mathcal{F}_v(\varphi_v). \tag{3.4}$$

3.3. **First and second term identities.** In this section, we recall certain identities relating the terms in the Laurent expansions of regularized theta integrals to those of Siegel Eisenstein series, which were proved by Gan, Qiu, and Takeda in [GQT14].

Let $n = 4$. Let $\mathcal{R}$ be the space of automorphic forms on $\mathrm{GSp}_8(\mathbb{A})$ spanned by residues $E^{(4,4)}_{-1}(1/2, F)$ for all holomorphic sections F of $I_{4,4}(s)$. Define a $(\mathfrak{g}_\infty, K_\infty) \times \mathrm{GSp}_8(\mathbb{A}_f)$-intertwining map

$$S(V^4_{3,3}(\mathbb{A})) \longrightarrow I_{4,4}\left(\frac{1}{2}\right), \quad \varphi \longmapsto F(\varphi)$$

by

$$F(\varphi)\left(g, \frac{1}{2}\right) = \omega(g,g')\varphi(0),$$

where $\nu(g') = \nu(g)$. We extend $F(\varphi)$ to the holomorphic section $F(\varphi)$ of $I_{4,4}(s)$ such that its restriction to $\mathbf{K}$ is independent of s.

As in [Ike96, §5], define a $(\mathfrak{g}_\infty, K_\infty) \times \mathrm{Sp}_8(\mathbb{A}_f)$-intertwining map

$$\mathrm{pr} : S(V_{3,3}^4(\mathbb{A})) \longrightarrow S(V_{2,2}^4(\mathbb{A}))$$

by

$$\mathrm{pr}(\varphi)\begin{pmatrix} x \\ y \end{pmatrix} = \int_{\mathbb{A}^4} \varphi_{K'}\begin{pmatrix} u \\ x \\ 0 \\ y \end{pmatrix} du$$

for $x, y \in \mathrm{M}_{2,4}(\mathbb{A})$, where du is the Tamagawa measure on $\mathbb{A}^4$. Then, as a special case of the Siegel-Weil formula proved by Gan, Qiu, and Takeda [GQT14, Theorem 1.1], we have

$$E_{-1}^{(4,4)}\left(\frac{1}{2}, F(\varphi)\right) = I_{-2}^{(4,3)}(\varphi), \tag{3.5}$$

$$E_{0}^{(4,4)}\left(\frac{1}{2}, F(\varphi)\right) = I_{-1}^{(4,3)}(\varphi) - \kappa_{3,2}\cdot I_{0}^{(4,2)}(\mathrm{pr}(\varphi)) \bmod \mathcal{R} \tag{3.6}$$

for all $\varphi \in S(V_{3,3}^4(\mathbb{A}))$.

4. Automorphic L-functions

In this section, we introduce integral representations of two automorphic L-functions, namely the standard L-function for GSp_4 and the Rankin-Selberg L-function for $\mathrm{GSp}_4 \times \mathrm{GSp}_4$. We keep the notation of § 3 with $n = 4$.

4.1. **Preliminaries.** Recall that $G = \mathrm{GSp}_4$. Let $H = \mathrm{GSp}_8$ and

$$\mathbf{G} = \{(g_1, g_2) \in G \times G \mid \nu(g_1) = \nu(g_2)\}.$$

Let Z_H be the center of H. We identify $\mathbf{G}$ with its image under the embedding

$$\mathbf{G} \longrightarrow H, \quad \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \longmapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & -b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & -c_2 & 0 & d_2 \end{pmatrix}.$$

Let π be an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character and paramodular conductor $\mathfrak{n}$ satisfying the conditions in § 2.4. When $v \nmid \infty\mathfrak{n}$ or $v \in S(\mathrm{PS})$, there exist $\lambda_{1,v}, \lambda_{2,v} \in \mathbb{C}$ such that

$$\pi_v|_{\mathrm{Sp}_4(k_v)} = \mathrm{Ind}_{\mathrm{Sp}_4(k_v)\cap \mathbf{B}(k_v)}^{\mathrm{Sp}_4(k_v)}(|\ |_v^{\lambda_{1,v}} \boxtimes |\ |_v^{\lambda_{2,v}}). \tag{4.1}$$

When $v \mid \mathfrak{n}$, there exist $\varepsilon_v \in \{0, 1\}$ and $\lambda_v \in \mathbb{C}$ such that

$$\pi_v = \mathrm{Ind}_{P_{2,2}(k_v)}^{G(k_v)}\left((\mathrm{St}_v \otimes |\ |_v^{\lambda_v}) \boxtimes \eta_v^{\varepsilon_v} |\ |_v^{-\lambda_v}\right), \tag{4.2}$$

where St_v is the Steinberg representation of $\mathrm{GL}_2(k_v)$ and η_v is the non-trivial unramified quadratic character of $k_v^\times$. Indeed, by the results in [RS07], any irreducible generic admissible representation of $G(k_v)$ with trivial central character and paramodular conductor $\varpi_v\mathfrak{o}_v$ is of this form. Since π_v is unitary and generic for all v, by the unitarizability criterion in [LMT04, Theorem 1.1], we have

$$\begin{cases} |\mathrm{Re}(\lambda_{1,v})| + |\mathrm{Re}(\lambda_{2,v})| < 1 & \text{if } v \nmid \infty\mathfrak{n}, \\ |\mathrm{Re}(\lambda_v)| < 1/2 & \text{if } v \mid \mathfrak{n}, \\ |\mathrm{Re}(\lambda_{1,v})| + |\mathrm{Re}(\lambda_{2,v})| < 1 & \text{if } v \in S(\mathrm{PS}). \end{cases} \tag{4.3}$$

4.2. **Standard L-functions.** In this section, we review the doubling method of Piatetski-Shapiro and Rallis [PSR87], [Har93, §6.2].

Let $P = P_{4,4}$ be the standard Siegel parabolic subgroup of H, and put

$$d_P(s) = \xi\left(s+\frac{5}{2}\right)\xi(2s+1)\xi(2s+3).$$

Let $I(s) = I_{4,4}(s)$ denote a degenerate principal series representation of $H(\mathbb{A})$. For a holomorphic section F of $I(s)$, we write $E(s,F) = E^{(4,4)}(s,F)$. By [KR94, Theorem 1.1], the Eisenstein series $E(s,F)$ has at most a simple pole at $s = 1/2$.

For $f \in \pi$, let

$$Z(s,f,F) = \int_{Z_H(\mathbb{A})\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A})} E(g;s,F)(f\otimes\bar{f})(g)\,dg.$$

We assume that $f = \bigotimes_v f_v$ and $F = \bigotimes_v F_v$. Choose local Hermitian pairings $\langle\ ,\ \rangle_v$ on $\pi_v \times \pi_v$ such that $\langle f,f\rangle = \prod_v \langle f_v, f_v\rangle_v$, and define a matrix coefficient ϕ_v of π_v by

$$\phi_v(g) = \frac{\langle \pi_v(g)f_v, f_v\rangle_v}{\langle f_v, f_v\rangle_v}.$$

Note that ϕ_v does not depend on the choice of $\langle\ ,\ \rangle_v$. Define a local zeta integral $Z_v(s,\phi_v,F_v)$ by

$$Z_v(s,\phi_v,F_v) = \int_{\mathrm{Sp}_4(k_v)} F_v(\delta(g_v,1),s)\phi_v(g_v)\,dg_v^{\mathrm{Tam}}, \tag{4.4}$$

where

$$\delta = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\mathbf{1}_2 & \frac{1}{2}\mathbf{1}_2 \\ \frac{1}{2}\mathbf{1}_2 & \frac{1}{2}\mathbf{1}_2 & 0 & 0 \\ \mathbf{1}_2 & -\mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_2 & \mathbf{1}_2 \end{pmatrix}.$$

Here dg_v^{Tam} is the local Tamagawa measure on $\mathrm{Sp}_4(k_v)$. Then, by [PSR87], [Har93, §6.2],

$$Z(s,f,F) = \langle f,f\rangle \prod_v Z_v(s,\phi_v,F_v)$$

for $\mathrm{Re}(s) \gg 0$.

Lemma 4.1. *Let v be a finite place of k satisfying the following conditions:*

- *π_v is unramified.*
- *f_v is $G(\mathfrak{o}_v)$-invariant.*
- *F_v is $H(\mathfrak{o}_v)$-invariant.*

We have

$$Z_v(s,\phi_v,F_v) = F_v(1,s)q_v^{-5\mathfrak{c}_v}\zeta_v(2)^{-1}\zeta_v(4)^{-1}d_{P,v}(s)^{-1}L\left(s+\frac{1}{2},\pi_v,\mathrm{std}\right).$$

Proof. Let dg_v^{std} be the Haar measure on $\mathrm{Sp}_4(k_v)$ normalized so that

$$\mathrm{vol}(\mathrm{Sp}_4(\mathfrak{o}_v), dg_v^{\mathrm{std}}) = 1.$$

By [PSR87, Proposition 6.2],

$$\int_{\mathrm{Sp}_4(k_v)} F_v(\delta(g_v,1),s)\phi_v(g_v)\,dg_v^{\mathrm{std}} = F_v(1,s)d_{P,v}(s)^{-1}L\left(s+\frac{1}{2},\pi_v,\mathrm{std}\right).$$

Since

$$\mathrm{vol}(\mathrm{Sp}_4(\mathfrak{o}_v), dg_v^{\mathrm{Tam}}) = q_v^{-5\mathfrak{c}_v}\cdot q_v^{-10}\cdot|\mathrm{Sp}_4(\mathbb{F}_{q_v})| = q_v^{-5\mathfrak{c}_v}\zeta_v(2)^{-1}\zeta_v(4)^{-1},$$

it follows that

$$dg_v^{\mathrm{Tam}} = q_v^{-5\mathfrak{c}_v}\zeta_v(2)^{-1}\zeta_v(4)^{-1}\cdot dg_v^{\mathrm{std}}.$$

This completes the proof. □

Lemma 4.2. *Let v be a place of k. The integral $Z_v(s,\phi_v,F_v)$ is absolutely convergent for*

$$\begin{cases} \mathrm{Re}(s) > -1/2 + \max\{|\mathrm{Re}(\lambda_{1,v})|, |\mathrm{Re}(\lambda_{2,v})|\} & \text{if } v \nmid \infty\mathfrak{n}, \\ \mathrm{Re}(s) > -1/2 + \max\{0, |\mathrm{Re}(\lambda)| - 1/2\} & \text{if } v \in S(\mathfrak{n}), \\ \mathrm{Re}(s) > -1/2 & \text{if } v \in S(\mathrm{DS}), \\ \mathrm{Re}(s) > -1/2 + \max\{|\mathrm{Re}(\lambda_{1,v})|, |\mathrm{Re}(\lambda_{2,v})|\} & \text{if } v \in S(\mathrm{PS}). \end{cases}$$

In particular, the integral is absolutely convergent for $\mathrm{Re}(s) \geq 1/2$.

Proof. The assertions will be proved in Lemmas 9.1-9.3 below. □

4.3. ***L*-functions for $\mathrm{GSp}_4 \times \mathrm{GSp}_4$.** In this section, we review Jiang's integral representation of the L-function for $G \times G$ [Jia96].

Let $\mathcal{P} = P_{4,3}$ be a maximal parabolic subgroup of H, and put

$$d_{\mathcal{P}}(s) = \xi(s+1)\xi(s+2)\xi(s+3)\xi(2s+2).$$

Let $\mathcal{I}(s) = I_{4,3}(s)$ denote a degenerate principal series representation of $H(\mathbb{A})$. For a holomorphic section $\mathcal{F}$ of $\mathcal{I}(s)$, we write $\mathcal{E}(s,\mathcal{F}) = E^{(4,3)}(s,\mathcal{F})$. By [Jia96, Chapter 3, Theorem 4.0.1], the Eisenstein series $\mathcal{E}(s,\mathcal{F})$ has at most a double pole at $s = 1$.

For $f \in \pi$, let

$$\mathcal{Z}(s,f,\mathcal{F}) = \int_{Z_H(\mathbb{A})\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A})} \mathcal{E}(g;s,\mathcal{F})(f \otimes \bar{f})(g)\,dg.$$

We assume that $f = \bigotimes_v f_v$ and $\mathcal{F} = \bigotimes_v \mathcal{F}_v$. Let $W = \bigotimes_v W_v$ be the Whittaker function of f. Define a local zeta integral $\mathcal{Z}_v(s,W_v,\mathcal{F}_v)$ by

$$\mathcal{Z}_v(s,W_v,\mathcal{F}_v) = \int_{Z_H(k_v)\tilde{U}(k_v)\backslash\mathbf{G}(k_v)} \mathcal{F}_v(\eta g_v,s)(W_v \otimes \overline{W_v})(g_v)\,d\bar{g}_v^{\mathrm{Tam}}, \tag{4.5}$$

where

$$\tilde{U} = \left\{ \left(u \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, u \right) \,\middle|\, u \in U,\, x \in \mathbb{G}_a \right\}$$

and

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here $d\bar{g}_v^{\mathrm{Tam}}$ is the quotient measure defined by the local Tamagawa measures on $Z_H(k_v)\backslash\mathbf{G}(k_v)$ and $\tilde{U}(k_v)$. Then, by [Jia96],

$$\mathcal{Z}(s,f,\mathcal{F}) = \prod_v \mathcal{Z}_v(s,W_v,\mathcal{F}_v)$$

for $\mathrm{Re}(s) \gg 0$.

Lemma 4.3. *Let v be a finite place of k satisfying the following conditions:*

- *π_v is unramified.*
- *W_v is $G(\mathfrak{o}_v)$-invariant.*
- *$\mathcal{F}_v$ is $H(\mathfrak{o}_v)$-invariant.*

We have

$$\begin{aligned}\mathcal{Z}_v(s,W_v,\mathcal{F}_v) &= \mathcal{F}_v(1,s)|W_v(\mathrm{diag}(\varpi_v^{-\mathfrak{c}_v},1,\varpi_v^{2\mathfrak{c}_v},\varpi_v^{\mathfrak{c}_v}))|^2 \\ &\quad \times q_v^{(3s/2-13)\mathfrak{c}_v}\zeta_v(2)^{-2}\zeta_v(4)^{-2}d_{\mathcal{P},v}(s)^{-1}L\left(\frac{s+1}{2},\pi_v \times \pi_v^\vee\right).\end{aligned}$$

Proof. Let

$$g_0 = \mathrm{diag}(\varpi_v^{\mathfrak{c}_v}, 1, \varpi_v^{-2\mathfrak{c}_v}, \varpi_v^{-\mathfrak{c}_v}).$$

Let $\psi'_v(x) = \psi_v(\varpi_v^{-\mathfrak{c}_v} x)$ be an additive character of k_v of conductor $\mathfrak{o}_v$. Let W'_v be the $G(\mathfrak{o}_v)$-invariant Whittaker function of π_v with respect to ψ'_v such that $W'_v(1) = 1$. Then

$$W_v(g) = W_v(g_0^{-1}) W'_v(g_0 g)$$

for $g \in G(k_v)$. Let dg_v^{std} and du_v^{std} be the Haar measures on $Z_H(k_v)\backslash \mathbf{G}(k_v)$ and $\tilde{U}(k_v)$ normalized so that

$$\mathrm{vol}(Z_H(\mathfrak{o}_v)\backslash G(\mathfrak{o}_v), dg_v^{\mathrm{std}}) = \mathrm{vol}(\tilde{U}(\mathfrak{o}_v), du_v^{\mathrm{std}}) = 1.$$

Let $d\bar{g}_v^{\mathrm{std}}$ be the corresponding quotient measure on $Z_H(k_v)\tilde{U}(k_v)\backslash \mathbf{G}(k_v)$. By [Jia96, Theorem 3.3.3],

$$\int_{Z_H(k_v)\tilde{U}(k_v)\backslash \mathbf{G}(k_v)} \mathcal{F}_v(\eta g_v, s)(W'_v \otimes \overline{W'_v})(g_v)\, d\bar{g}_v^{\mathrm{std}} = \mathcal{F}_v(1,s) d_{\mathcal{P},v}(s)^{-1} L\left(\frac{s+1}{2}, \pi_v \times \pi_v^\vee\right).$$

On the other hand,

$$\mathcal{F}_v(\eta(g_0^{-1}, g_0^{-1})g, s) = q_v^{(3s/2+9/2)\mathfrak{c}_v} \mathcal{F}_v(\eta g, s)$$

for $g \in \mathbf{G}(k_v)$. Therefore,

$$\begin{aligned}
&\int_{Z_H(k_v)\tilde{U}(k_v)\backslash \mathbf{G}(k_v)} \mathcal{F}_v(\eta g_v, s)(W_v \otimes \overline{W_v})(g_v)\, d\bar{g}_v^{\mathrm{std}} \\
&= q_v^{-10\mathfrak{c}_v} |W_v(g_0^{-1})|^2 \int_{Z_H(k_v)\tilde{U}(k_v)\backslash \mathbf{G}(k_v)} \mathcal{F}_v(\eta(g_0^{-1}, g_0^{-1})g_v, s)(W'_v \otimes \overline{W'_v})(g_v)\, d\bar{g}_v^{\mathrm{std}} \\
&= q_v^{(3s/2-11/2)\mathfrak{c}_v} |W_v(g_0^{-1})|^2 \mathcal{F}_v(1,s) d_{\mathcal{P},v}(s)^{-1} L\left(\frac{s+1}{2}, \pi_v \times \pi_v^\vee\right).
\end{aligned}$$

It suffices to show that

$$d\bar{g}_v^{\mathrm{Tam}} = q_v^{-15\mathfrak{c}_v/2} \zeta_v(2)^{-2} \zeta_v(4)^{-2} \cdot d\bar{g}_v^{\mathrm{std}}.$$

Let dg_v^{Tam}, $dg_{1,v}^{\mathrm{Tam}}$, and $dg_{2,v}^{\mathrm{Tam}}$ be the local Tamagawa measures on $Z_H(k_v)\backslash \mathbf{G}(k_v)$, $\mathrm{PGSp}_4(k_v)$, and $\mathrm{Sp}_4(k_v)$, respectively. Under the isomorphism

$$\begin{aligned}
\mathrm{PGSp}_4(k_v) \ltimes \mathrm{Sp}_4(k_v) &\longrightarrow Z_H(k_v)\backslash \mathbf{G}(k_v), \\
(g_1, g_2) &\longmapsto (g_1 g_2, g_1),
\end{aligned}$$

we have

$$dg_{1,v}^{\mathrm{Tam}} \times dg_{2,v}^{\mathrm{Tam}} = dg_v^{\mathrm{Tam}}.$$

Let

$$U' = \left\{ \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \ x \in \mathbb{G}_a \right\}$$

be a unipotent subgroup of Sp_4. Let du_v^{Tam}, $du_{1,v}^{\mathrm{Tam}}$, and $du_{2,v}^{\mathrm{Tam}}$ be the local Tamagawa measures on $\tilde{U}(k_v)$, $U(k_v)$, and $U'(k_v)$, respectively. Then

$$U(k_v) \ltimes U'(k_v) \simeq \tilde{U}(k_v)$$

under the above isomorphism and

$$du_{1,v}^{\mathrm{Tam}} \times du_{2,v}^{\mathrm{Tam}} = du_v^{\mathrm{Tam}}.$$

Note that

$$\begin{aligned}
\mathrm{vol}(Z_H(\mathfrak{o}_v)\tilde{U}(\mathfrak{o}_v)\backslash \mathbf{G}(\mathfrak{o}_v), d\bar{g}_v^{\mathrm{Tam}}) &= \frac{\mathrm{vol}(\mathrm{PGSp}_4(\mathfrak{o}_v), dg_{1,v}^{\mathrm{Tam}})}{\mathrm{vol}(U(\mathfrak{o}_v), du_{1,v}^{\mathrm{Tam}})} \cdot \frac{\mathrm{vol}(\mathrm{Sp}_4(\mathfrak{o}_v), dg_{2,v}^{\mathrm{Tam}})}{\mathrm{vol}(U'(\mathfrak{o}_v), du_{2,v}^{\mathrm{Tam}})} \\
&= \frac{q_v^{-5\mathfrak{c}_v} \cdot q_v^{-10} \cdot |\mathrm{PGSp}_4(\mathbb{F}_{q_v})|}{q_v^{-2\mathfrak{c}_v}} \cdot \frac{q_v^{-5\mathfrak{c}_v} \cdot q_v^{-10} \cdot |\mathrm{Sp}_4(\mathbb{F}_{q_v})|}{q_v^{-\mathfrak{c}_v/2}} \\
&= q_v^{-15\mathfrak{c}_v/2} \zeta_v(2)^{-2} \zeta_v(4)^{-2}.
\end{aligned}$$

This completes the proof. □

Lemma 4.4. *Let v be a place of k such that $v \mid \infty\mathfrak{n}$. The integral $\mathcal{Z}_v(s, W_v, \mathcal{F}_v)$ is absolutely convergent for*

$$\begin{cases} \mathrm{Re}(s) > -1 + 4|\mathrm{Re}(\lambda_v)| & \text{if } v \in S(\mathfrak{n}), \\ \mathrm{Re}(s) > -1 & \text{if } v \in S(\mathrm{DS}), \\ \mathrm{Re}(s) > -1 + 2(|\mathrm{Re}(\lambda_{1,v})| + |\mathrm{Re}(\lambda_{2,v})|) & \text{if } v \in S(\mathrm{PS}), \end{cases}$$

In particular, the integral is absolutely convergent for $\mathrm{Re}(s) \geq 1$.

Proof. The assertion will be proved in Lemma 9.5 below. □

5. Proof of Main Theorem

5.1. **Crude form of the formula for Petersson norms.** In this section, we keep the notation of § 3 and § 4. Let π be an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character and paramodular conductor $\mathfrak{n}$ satisfying the conditions in § 2.4. Let $f \in \pi$ be the cusp form satisfying the conditions (2.4) and (2.5). Let S be the set of places of k defined by

$$S = \{v \mid \infty\mathfrak{n}\}.$$

For an automorphic form ϕ on $H(\mathbb{A}) = \mathrm{GSp}_8(\mathbb{A})$, let

$$\langle \phi, \bar{f} \otimes f \rangle = \int_{Z_H(\mathbb{A})\mathbf{G}(k)\backslash \mathbf{G}(\mathbb{A})} \phi(g)(f \otimes \bar{f})(g)\, dg.$$

Lemma 5.1. *Assume that π is stable. For all $\phi \in \mathcal{R}$,*

$$\langle \phi, \bar{f} \otimes f \rangle = 0.$$

Proof. By Lemma 4.1, we have

$$\langle E(s,F), \bar{f} \otimes f \rangle = \langle f, f \rangle \cdot \xi^T(2)^{-1}\xi^T(4)^{-1} \cdot d_P^T(s)^{-1} \cdot L^T\left(s + \frac{1}{2}, \pi, \mathrm{std}\right) \prod_{v \in T} Z_v(s, \phi_v, F_v)$$

for any holomorphic section $F = \bigotimes_v F_v$ such that $F_v(1,s) = 1$ for all $v \notin T$, where T is a sufficiently large finite set of places of k depending on F. Since π is stable and hence $L^T(s, \pi, \mathrm{std})$ is holomorphic at $s = 1$, this together with Lemma 4.2 implies that $\langle E(s,F), \bar{f} \otimes f \rangle$ is holomorphic at $s = 1$. This completes the proof. □

Lemma 5.2. *Assume that π is stable. For all $d \in \mathbb{Z}$ and all $\varphi \in S(V_{2,2}^4(\mathbb{A}))$,*

$$\langle I_d^{(4,2)}(\varphi), \bar{f} \otimes f \rangle = 0.$$

Proof. Let $\theta_{r,r}(\pi)$ be the global theta lift of π to $\mathrm{GO}_{r,r}(\mathbb{A})$. By (4.1), (4.3), and the local theta correspondence for unramified representations, we have $\theta_{1,1}(\pi) = 0$. If $\theta_{2,2}(\pi) \neq 0$, then there exist irreducible cuspidal automorphic representations σ_1 and σ_2 of $\mathrm{GL}_2(\mathbb{A})$ such that

$$L^T(s, \pi, \mathrm{std}) = \xi^T(s) L^T(s, \sigma_1 \times \sigma_2),$$

where T is a sufficiently large finite set of places of k. This contradicts the holomorphy of $L^T(s, \pi, \mathrm{std})$ at $s = 1$. Hence

$$\theta_{2,2}(\pi) = 0. \tag{5.1}$$

Write $G' = \mathrm{GO}_{2,2}$ and $G_1' = \mathrm{O}_{2,2}$. Let $\mathcal{C} = \mathbb{A}^{\times,2}k^\times\backslash\mathbb{A}^\times$. The similitude characters induce isomorphisms

$$Z_G(\mathbb{A})\,\mathrm{Sp}_4(\mathbb{A})G(k)\backslash G(\mathbb{A}) \simeq \mathcal{C}, \quad Z_{G'}(\mathbb{A})G_1'(\mathbb{A})G'(k)\backslash G'(\mathbb{A}) \simeq \mathcal{C}.$$

Fix cross-sections $c \mapsto g_c$ and $c \mapsto g_c'$ of $G(\mathbb{A}) \to \mathcal{C}$ and $G'(\mathbb{A}) \to \mathcal{C}$, respectively. For $\varphi \in S(V_{2,2}^4(\mathbb{A}))$, write $z \cdot \varphi = \sum_j \varphi_{1j} \otimes \overline{\varphi_{2j}} \in S(V_{2,2}^4(\mathbb{A}))$ with $\varphi_{ij} \in S(V_{2,2}^2(\mathbb{A}))$, where z is the regularizing differential operator as in § 3.2. Then

$$\Theta((g_1 g_c, g_2 g_c), g' g_c'; z \cdot \varphi) = \sum_j \Theta(g_1 g_c, g' g_c'; \varphi_{1j}) \overline{\Theta(g_2 g_c, g' g_c'; \varphi_{2j})}$$

for $c \in \mathcal{C}$, $(g_1, g_2) \in \mathrm{Sp}_4(\mathbb{A}) \times \mathrm{Sp}_4(\mathbb{A})$, and $g' \in G_1'(\mathbb{A})$. By (5.1), the theta lift $\theta(f, \varphi_{ij})$ is identically zero, where

$$\theta(g' g_c'; f, \varphi_{ij}) = \int_{\mathrm{Sp}_4(k)\backslash \mathrm{Sp}_4(\mathbb{A})} \Theta(g g_c, g' g_c'; \varphi_{ij}) f(g)\, dg$$

for $c \in \mathcal{C}$ and $g' \in G_1'(\mathbb{A})$. Hence $\langle I^{(4,2)}(s,\varphi), \bar{f} \otimes f \rangle$ is equal to

$$\int_{Z_H(\mathbb{A})\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A})} I^{(4,2)}(g;s,\varphi)(f\otimes\bar{f})(g)\,dg$$
$$= \kappa_{2,0}^{-1} Q_{4,2}(s)^{-1} \int_{\mathcal{C}} \int_{(\mathrm{Sp}_4(k)\backslash \mathrm{Sp}_4(\mathbb{A}))^2} \int_{G_1'(k)\backslash G_1'(\mathbb{A})} \Theta((g_1 g_c, g_2 g_c), g' g_c'; z\cdot\varphi) E(g';s) f(g_1 g_c)\overline{f(g_2 g_c)}\,dg'\,dg_1\,dg_2\,dc$$
$$= \kappa_{2,0}^{-1} Q_{4,2}(s)^{-1} \int_{Z_{G'}(\mathbb{A})G'(k)\backslash G'(\mathbb{A})} \sum_j \theta(g';f,\varphi_{1j})\overline{\theta(g';f,\varphi_{2j})} E(g';s)\,dg'$$
$$= 0.$$

This completes the proof. $\square$

Lemma 5.3. *Let $v \in S$. There exists $\varphi_v \in S(V_{3,3}^4(k_v))$ such that*

$$Z_v\left(\frac{1}{2}, \phi_v, F_v(\varphi_v)\right) \neq 0.$$

Moreover, for $v \mid \mathfrak{n}$ or $v \in S(\mathrm{PS})$, then we can find such φ_v which does not depend on π_v.

Proof. We identify $k_v = \mathbb{R}$ if v is a real place. Let π_0 be an irreducible component of $\pi_v|_{\mathrm{Sp}_4(k_v)}$. Fix a pairing $\langle\ ,\ \rangle$ on π_0. For $f_1, f_2 \in \pi_0$, define a matrix coefficient $\phi_{f_1\otimes f_2}$ of π_0 by

$$\phi_{f_1\otimes f_2}(g) = \langle \pi_0(g) f_1, f_2 \rangle.$$

Then the local zeta integral $Z_v(1/2, \phi_{f_1\otimes f_2}, F)$ is absolutely convergent by Lemma 4.2, and defines an $\mathrm{Sp}_4(k_v)\times\mathrm{Sp}_4(k_v)$-intertwining map (resp. $(\mathfrak{sp}_4(\mathbb{R}), \mathrm{U}(2)) \times (\mathfrak{sp}_4(\mathbb{R}), \mathrm{U}(2))$-intertwining map)

$$I_{4,4,v}\left(\frac{1}{2}\right) \longrightarrow \pi_0^\vee \otimes \pi_0, \quad F \longmapsto \left[f_1 \otimes f_2 \mapsto Z_v\left(\frac{1}{2}, \phi_{f_1\otimes f_2}, F\right)\right]$$

if v is finite (resp. v is real). By [KR90b, Corollary 3.2.3] and [KR94, Proposition 7.2.1], for fixed non-zero elements $f_1, f_2 \in \pi_0$, there exists $F \in I_{4,4,v}(1/2)$ such that

$$Z_v\left(\frac{1}{2}, \phi_{f_1\otimes f_2}, F\right) \neq 0. \tag{5.2}$$

Moreover, if $v \mid \mathfrak{n}$, then by the proof of [KR94, Proposition 7.2.1], we can find such F which depends only on the stabilizers of f_1 and f_2 in $\mathrm{Sp}_4(k_v)$.

Assume $v \mid \mathfrak{n}$. Let V_0 be the split quadratic space of dimension 4 over k_v. Let V_1 (resp. V_2) be the split quadratic space (resp. quaternionic quadratic space) of dimension 6 over k_v and $\omega_{\psi_v,V_i,4}$ the Weil representation of $\mathrm{Sp}_8(k_v)\times\mathrm{O}(V_i)(k_v)$ on $S(V_i^4(k_v))$ with respect to ψ_v. For $i=1,2$, let $R(V_i)$ be the image of the $\mathrm{Sp}_8(k_v)$-intertwining map

$$S(V_i^4(k_v)) \longrightarrow I_{4,4,v}\left(\frac{1}{2}\right), \quad \varphi \longmapsto F_v(\varphi),$$

where $F_v(\varphi)(g,\frac{1}{2}) = \omega_{\psi_v,V_i,4}(g,1)\varphi(0)$. By [KR92],

$$I_{4,4,v}\left(\frac{1}{2}\right) = R(V_1) + R(V_2). \tag{5.3}$$

For $i=0,1,2$, let $\theta_i(\pi_0^\vee)$ be the theta lift of $\pi_0^\vee$ to $\mathrm{O}(V_i)(k_v)$. By our assumption on π_0 and [GI11, Theorem A.10], $\theta_0(\pi_0^\vee)\neq 0$. Therefore, by the conservation relation [SZ15] (see also [KR05, Theorem 3.8]), $\theta_2(\pi_0^\vee)=0$. By [HKS96, Proposition 3.1], this is equivalent to

$$\mathrm{Hom}_{\mathrm{Sp}_4(k_v)\times\mathrm{Sp}_4(k_v)}(R(V_2), \pi_0^\vee\otimes\pi_0) = 0.$$

From this together with (5.2) and (5.3), we deduce that $Z_v(1/2, \phi_{f_1\otimes f_2}, F)\neq 0$ for some $F\in R(V_1)$ which depends only on the stabilizers of f_1 and f_2 in $\mathrm{Sp}_4(k_v)$. This completes the proof for $v\mid\mathfrak{n}$.

Assume $v \in S(\mathrm{DS})$. For non-negative integers p,q, let $V_{p,q}$ denote the quadratic space over $\mathbb{R}$ of signature (p,q) and $\omega_{\psi_v,V_{p,q},4}$ the Weil representation of $\mathrm{Sp}_8(\mathbb{R})\times\mathrm{O}_{p,q}(\mathbb{R})$ on $S(V_{p,q}^4(\mathbb{R}))$ with respect to ψ_v. When p,

q are positive odd integers such that $p+q=6$, let $R(p,q)$ be the image of the $(\mathfrak{sp}_8(\mathbb{R}), \mathrm{U}(4))$-intertwining map

$$S(V^4_{p,q}(\mathbb{R})) \longrightarrow I_{4,4,v}\left(\frac{1}{2}\right), \quad \varphi \longmapsto F_v(\varphi),$$

where $F_v(\varphi)(g, \frac{1}{2}) = \omega_{\psi_v, V_{p,q}, 4}(g,1)\varphi(0)$. By [KR90a], [LZ97],

$$I_{4,4,v}\left(\frac{1}{2}\right) = R(5,1) + R(3,3) + R(1,5). \tag{5.4}$$

Let $\theta_{p,q}(\pi_0^\vee)$ be the theta lift of $\pi_0^\vee$ to $\mathrm{O}_{p,q}(\mathbb{R})$. By our assumption on π_0 and [Pau05, Theorem 18], $\theta_{2,2}(\pi_0^\vee) \neq 0$. Therefore, by the conservation relation [SZ15] (see also [Pau05, Proposition 22]), $\theta_{5,1}(\pi_0^\vee) = \theta_{1,5}(\pi_0^\vee) = 0$. Hence, as in the proof of [HKS96, Proposition 3.1],

$$\mathrm{Hom}_{(\mathfrak{sp}_4(\mathbb{R}),\mathrm{U}(2)) \times (\mathfrak{sp}_4(\mathbb{R}),\mathrm{U}(2))}(R(5,1), \pi_0^\vee \otimes \pi_0) = 0,$$
$$\mathrm{Hom}_{(\mathfrak{sp}_4(\mathbb{R}),\mathrm{U}(2)) \times (\mathfrak{sp}_4(\mathbb{R}),\mathrm{U}(2))}(R(1,5), \pi_0^\vee \otimes \pi_0) = 0.$$

From this together with (5.2) and (5.4), we deduce that $Z_v(1/2, \phi_{f_1 \otimes f_2}, F) \neq 0$ for some $F \in R(3,3)$. This completes the proof for $v \in S(\mathrm{DS})$.

Assume $v \in S(\mathrm{PS})$. Let $\varphi_v \in S(V^4_{3,3}(\mathbb{R}))$ be the Gaussian function. Proceeding similarly as in the proof of [PSR87, Proposition 6.2] or Lemma 8.11 below, we see that

$$Z_v\left(\frac{1}{2}, \phi_v, F_v(\varphi_v)\right) \neq 0.$$

This completes the proof. □

Proposition 5.4. *We have*

$$\langle f, f \rangle = 2^c \cdot \frac{L(1,\pi,\mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot \prod_v C(\pi_v).$$

Here $C(\pi_v)$ is a non-zero constant depending only on π_v for each place v, and

$$\Delta_{\mathrm{PGSp}_4} = \xi(2)\xi(4),$$

$$c = \begin{cases} 1 & \text{if } \pi \text{ is stable}, \\ 2 & \text{if } \pi \text{ is endoscopic}. \end{cases}$$

In fact, $C(\pi_v) = q_v^{-5\mathfrak{c}_v}$ if $v \notin S$ and

$$C(\pi_v) = \zeta_v(1)^{-1}\zeta_v(3)^{-1}\zeta_v(4)L(1,\pi_v,\mathrm{Ad})^{-1} \cdot \frac{\mathcal{Z}_v(1, W_v, \mathcal{F}_v(\varphi_v))}{Z_v\left(1/2, \phi_v, F_v(\varphi_v)\right)} \cdot \begin{cases} q_v^{-9\mathfrak{c}_v/2} & \text{if } v \mid \mathfrak{n}, \\ 1 & \text{if } v \mid \infty, \end{cases} \tag{5.5}$$

where $\varphi_v \in S(V^4_{3,3}(k_v))$ is any Schwartz function such that $Z_v(1/2, \phi_v, F_v(\varphi_v)) \neq 0$.

Proof. Let $\varphi = \bigotimes_v \varphi_v \in S(V^4_{3,3}(\mathbb{A}))$ be such that $\varphi_v = \varphi_v^o$ for all places $v \notin S$. Then the holomorphic section $F(\varphi) = \bigotimes_v F_v(\varphi_v)$ of $I(s)$ satisfies the following conditions:

- $F_v(\varphi_v)$ is $H(\mathfrak{o}_v)$-invariant and $F_v(\varphi_v)(1,s) = q_v^{-6\mathfrak{c}_v}$ for all places $v \notin S$,
- $F_v(\varphi_v)$ depends only on φ_v for all places $v \in S$.

Also, the holomorphic section $\mathcal{F}(\varphi) = \mathfrak{D}^{-9/2}\rho^{-1}\xi(2)^{-1}\xi(3)^{-1} \cdot \bigotimes_v \mathcal{F}_v(\varphi_v)$ of $\mathcal{I}(s)$ for $\mathrm{Re}(s) > -1$ satisfies the following conditions:

- $\mathcal{F}_v(\varphi_v)$ is $H(\mathfrak{o}_v)$-invariant and $\mathcal{F}_v(\varphi_v)(1,s) = \zeta_v(s+1)\zeta_v(s+2)\zeta_v(s+3)$ for all places $v \notin S$,
- $\mathcal{F}_v(\varphi_v)$ depends only on φ_v for all places $v \in S$.

Let $T = \{v \mid \infty\mathfrak{n}\mathfrak{d}\}$. By Lemmas 4.1 and 4.3, we have

$$\langle E(s, F(\varphi)), \bar{f} \otimes f \rangle = \langle f, f \rangle \cdot \xi^T(2)^{-1}\xi^T(4)^{-1} \cdot d_P^T(s)^{-1} \cdot L^T\left(s + \frac{1}{2}, \pi, \mathrm{std}\right) \prod_{v \in T} Z_v(s, \phi_v, F_v(\varphi_v)),$$

$$\langle \mathcal{E}(s, \mathcal{F}(\varphi)), \bar{f} \otimes f \rangle = \mathfrak{D}^{-9/2}\rho^{-1}\xi(2)^{-1}\xi(3)^{-1} \cdot \xi^T(2)^{-2}\xi^T(4)^{-2} \cdot d_{\mathcal{P}}^T(s)^{-1} \cdot \xi^T(s+1)\xi^T(s+2)\xi^T(s+3)$$
$$\times L^T\left(\frac{s+1}{2}, \pi \times \pi^\vee\right) \prod_{v \in T} \mathcal{Z}_v(s, W_v, \mathcal{F}_v(\varphi_v)).$$

On the other hand, by the first term identity (3.5),

$$\langle \mathcal{E}_{-2}(1, \mathcal{F}(\varphi)), \bar{f} \otimes f \rangle = \left\langle E_{-1}\left(\frac{1}{2}, F(\varphi)\right), \bar{f} \otimes f \right\rangle.$$

Hence, noting that $d_P^T(1/2) = \xi^T(2)\xi^T(3)\xi^T(4)$, $d_{\mathcal{P}}^T(1) = \xi^T(2)\xi^T(3)\xi^T(4)^2$, and

$$L^T(s, \pi \times \pi^\vee) = \xi^T(s) L^T(s, \pi, \mathrm{std}) L^T(s, \pi, \mathrm{Ad}),$$

we have

$$\begin{aligned} &4\mathfrak{D}^{-9/2}\xi(2)^{-1}\xi(3)^{-1}\xi^T(4)^{-1} \cdot \mathrm{Res}_{s=1} L^T(s, \pi, \mathrm{std}) \cdot \frac{L^T(1, \pi, \mathrm{Ad})}{\xi^T(2)\xi^T(4)} \prod_{v \in T} \zeta_v(1)^{-1} \mathcal{Z}_v(1, W_v, \mathcal{F}_v(\varphi_v)) \\ &= \langle f, f \rangle \xi^T(2)^{-1}\xi^T(3)^{-1}\xi^T(4)^{-1} \cdot \mathrm{Res}_{s=1} L^T(s, \pi, \mathrm{std}) \prod_{v \in T} Z_v\left(\frac{1}{2}, \phi_v, F_v(\varphi_v)\right). \end{aligned} \tag{5.6}$$

If π is endoscopic, then since $\mathrm{Res}_{s=1} L^T(s, \pi, \mathrm{std}) \neq 0$ and $\prod_{v \in T} L(s, \pi_v, \mathrm{Ad})$ is holomorphic and non-zero at $s = 1$, we deduce that

$$\begin{aligned} &\langle f, f \rangle \prod_{v \in T} Z_v\left(\frac{1}{2}, \phi_v, F_v(\varphi_v)\right) \\ &= 4\mathfrak{D}^{-9/2} \cdot \frac{L(1, \pi, \mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \prod_{v \in T} \zeta_v(1)^{-1}\zeta_v(3)^{-1}\zeta_v(4) L(1, \pi_v, \mathrm{Ad})^{-1} \mathcal{Z}_v(1, W_v, \mathcal{F}_v(\varphi_v)). \end{aligned} \tag{5.7}$$

If π is stable, then $L^T(s, \pi, \mathrm{std})$ is holomorphic and non-zero at $s = 1$, so that both sides of (5.6) vanish. However, by the second term identity (3.6) and Lemmas 5.1, 5.2,

$$\langle \mathcal{E}_{-1}(1, \mathcal{F}(\varphi)), \bar{f} \otimes f \rangle = \left\langle E_0\left(\frac{1}{2}, F(\varphi)\right), \bar{f} \otimes f \right\rangle.$$

Hence, similarly as above, we have

$$\begin{aligned} &2\mathfrak{D}^{-9/2}\xi(2)^{-1}\xi(3)^{-1}\xi^T(4)^{-1} \cdot L^T(1, \pi, \mathrm{std}) \cdot \frac{L^T(1, \pi, \mathrm{Ad})}{\xi^T(2)\xi^T(4)} \prod_{v \in T} \zeta_v(1)^{-1} \mathcal{Z}_v(1, W_v, \mathcal{F}_v(\varphi_v)) \\ &= \langle f, f \rangle \xi^T(2)^{-1}\xi^T(3)^{-1}\xi^T(4)^{-1} \cdot L^T(1, \pi, \mathrm{std}) \prod_{v \in T} Z_v\left(\frac{1}{2}, \phi_v, F_v(\varphi_v)\right), \end{aligned}$$

and deduce that

$$\begin{aligned} &\langle f, f \rangle \prod_{v \in T} Z_v\left(\frac{1}{2}, \phi_v, F_v(\varphi_v)\right) \\ &= 2\mathfrak{D}^{-9/2} \cdot \frac{L(1, \pi, \mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \prod_{v \in T} \zeta_v(1)^{-1}\zeta_v(3)^{-1}\zeta_v(4) L(1, \pi_v, \mathrm{Ad})^{-1} \mathcal{Z}_v(1, W_v, \mathcal{F}_v(\varphi_v)). \end{aligned} \tag{5.8}$$

Now recall that $\varphi_v = \varphi_v^o$ and hence $Z_v(1/2, \phi_v, F_v(\varphi_v)) \neq 0$ for all $v \notin S$ by Lemma 4.1. Also, we may choose φ_v such that $Z_v(1/2, \phi_v, F_v(\varphi_v)) \neq 0$ for all $v \in S$ by Lemma 5.3. For each place v of k, let

$$C(\pi_v) = \zeta_v(1)^{-1}\zeta_v(3)^{-1}\zeta_v(4) L(1, \pi_v, \mathrm{Ad})^{-1} \cdot \frac{\mathcal{Z}_v(1, W_v, \mathcal{F}_v(\varphi_v))}{Z_v(1/2, \phi_v, F_v(\varphi_v))} \cdot \begin{cases} q_v^{-9\mathfrak{c}_v/2} & \text{if } v \text{ is finite,} \\ 1 & \text{if } v \text{ is real.} \end{cases}$$

Note that $C(\pi_v)$ is a purely local constant depending only on π_v and φ_v. By Lemma 4.1 and 4.3, $C(\pi_v) = q_v^{-5\mathfrak{c}_v}$ for all $v \notin S$. In particular, $C(\pi_v) = 1$ for all $v \notin T$. Finally, by (5.7) and (5.8), we have

$$\langle f, f \rangle = 2^c \cdot \frac{L(1, \pi, \mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot \prod_v C(\pi_v).$$

Since the left-hand side is non-zero and independent of φ_v, so is $C(\pi_v)$ for all $v \in S$. This completes the proof. □

In the following proposition, we give an explicit Rallis inner product formula.

Proposition 5.5. *Assume π is endoscopic. We have*

$$\langle f, f\rangle = 4\cdot\frac{L(1,\pi,\mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}}\cdot\prod_v C'(\pi_v),$$

where

$$C'(\pi_v) = \begin{cases} q_v^{-5\mathfrak{c}_v} & \text{if } v\notin S,\\ q_v^{-1-5\mathfrak{c}_v}\zeta_v(2)^{-1}\zeta_v(4) & \text{if } v\mid\mathfrak{n},\\ 2^{\lambda_{1,v}-\lambda_{2,v}+5}\pi^{3\lambda_{1,v}-\lambda_{2,v}+5}(1+\lambda_{1,v}-\lambda_{2,v})^{-1} & \text{if } v\in S(\mathrm{DS}),\\ 2^{-4} & \text{if } v\in S(\mathrm{PS}).\end{cases} \tag{5.9}$$

Proof. The assertion follows from Propositions 6.1 and 6.2, whose proofs will be given in § 7 and § 8. □

Corollary 5.6. *Assume π is endoscopic. We have*

$$\prod_{v\in S} C(\pi_v) = \prod_{v\in S} C'(\pi_v).$$

Proof. The assertion follows from Propositions 5.4 and 5.5. □

5.2. **Families of local representations of** GSp_4**.** In this section, we switch to a local setting. Let F be a local field of characteristic zero. When F is non-archimedean, let $\mathfrak{o}$, $\mathfrak{c}$, ϖ, q, $|\ |$ be the notation defined as in § 2.1, and fix a non-trivial additive character ψ of F with conductor $\varpi^{-\mathfrak{c}}\mathfrak{o}$. When $F=\mathbb{R}$, let $\psi(x)=e^{2\pi\sqrt{-1}x}$.

Consider the split orthogonal similitude group $\mathrm{GO}_{2,2}$. Put

$$H=\mathrm{GO}_{2,2},\quad H^\circ=\mathrm{GSO}_{2,2}.$$

There is an isomorphism (cf. § 6.1)

$$\mathbb{G}_m\backslash(\mathrm{GL}_2\times\mathrm{GL}_2)\simeq H^\circ.$$

We write $[h_1,h_2]\in H^\circ$ for the image of $(h_1,h_2)\in\mathrm{GL}_2\times\mathrm{GL}_2$ under the above isomorphism. Let $\mathbf{t}\in H\setminus H^\circ$ be an involution so that $\mathrm{Ad}(\mathbf{t})[h_1,h_2]=[\det(h_2)^{-1}h_2,\det(h_1)^{-1}h_1]$. Let σ_1 and σ_2 be irreducible admissible representations of $\mathrm{GL}_2(F)$ with central characters inverse to each other. Let $\mathcal{V}_{\sigma_i}$ be the space of σ_i for $i=1,2$. Let $\sigma_1\times\sigma_2$ be the representation of $H^\circ(F)$ on $\mathcal{V}_{\sigma_1}\otimes\mathcal{V}_{\sigma_2}$ defined by

$$[h_1,h_2]\cdot(v_1\otimes v_2)=\sigma_1(h_1)v_1\otimes\sigma_2(h_2)v_2$$

for $h_1,h_2\in\mathrm{GL}_2(F)$ and $v_1\in\mathcal{V}_{\sigma_1}$, $v_2\in\mathcal{V}_{\sigma_2}$. We define an irreducible admissible representation $(\sigma_1\times\sigma_2)^\sharp$ of $H(F)$ as follows:

- If $\sigma_1\not\simeq\sigma_2^\vee$, then $(\sigma_1\times\sigma_2)^\sharp=\mathrm{Ind}_{H^\circ(F)}^{H(F)}(\sigma_1\times\sigma_2)$.
- If $\sigma_1\simeq\sigma_2^\vee$, then since $\sigma_2^\vee\simeq\sigma_2\otimes\omega_{\sigma_2}^{-1}$, we may assume $\mathcal{V}_{\sigma_1}=\mathcal{V}_{\sigma_2}$. Let $\mathcal{V}^\sharp=\mathcal{V}_{\sigma_1}\otimes\mathcal{V}_{\sigma_1}$. Let $(\sigma_1\times\sigma_2)^\sharp$ be the representation of $H(F)$ on $\mathcal{V}^\sharp$ defined by

$$\begin{aligned}[h_1,h_2]\cdot(v_1\otimes v_2)&=\sigma_1(h_1)v_1\otimes\omega_{\sigma_1}(\det(h_2)^{-1})\sigma_1(h_2)v_2,\\ \mathbf{t}\cdot(v_1\otimes v_2)&=v_2\otimes v_1\end{aligned}$$

  for $h_1,h_2\in\mathrm{GL}_2(F)$ and $v_1,v_2\in\mathcal{V}_{\sigma_1}$.

We consider three types of generic irreducible admissible representations of $G(F)$ with trivial central character:

(IIa) F is non-archimedean and π is a representation of $G(F)$ with paramodular conductor $\varpi\mathfrak{o}$.
(DS) $F=\mathbb{R}$ and π is a (limit of) discrete series representation of $G(\mathbb{R})$.
(PS) $F=\mathbb{R}$ and π is a principal series representation of $G(\mathbb{R})$ with non-zero $(\mathrm{Sp}_4(\mathbb{R})\cap\mathrm{O}(4))$-invariant vectors.

Note that we do not require π to be unitary.

Let π be a representation of type (IIa). Then, as explained in the proof of [RS07, Proposition 7.2.5], π is induced from the following representation of the standard Siegel parabolic subgroup

$$\begin{pmatrix} A & * \\ 0 & \nu {}^tA^{-1}\end{pmatrix}\longmapsto(\mathrm{St}\otimes|\ |^{\lambda})(A)\cdot\eta^{\varepsilon}|\nu|^{-\lambda} \tag{5.10}$$

for some $\varepsilon \in \{0,1\}$ and $\lambda \in \mathbb{C}$. Here St is the Steinberg representation of $\mathrm{GL}_2(F)$ and η is the non-trivial unramified quadratic character of $F^\times$. Note that $\lambda \bmod \frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}$ is determined by π up to sign. We call ε and λ the sign and the parameter of π, respectively. Note that π is the theta lift of the representation

$$\left((\mathrm{St} \otimes \eta^\varepsilon) \times \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(|\ |^\lambda \eta^\varepsilon \boxtimes |\ |^{-\lambda}\eta^\varepsilon)\right)^\sharp \tag{5.11}$$

of $H(F)$ to $G(F)$.

Let π be a representation of type (DS). Then

$$\pi|_{\mathrm{Sp}_4(\mathbb{R})} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)}.$$

Here $D_{(\lambda_1,\lambda_2)}$ is the (limit of) discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ with Blattner parameter (λ_1,λ_2) such that $1-\lambda_1 \leq \lambda_2 \leq 0$. We call $\lambda = (\lambda_1,\lambda_2) \in \mathbb{Z}^2$ the parameter of π. Note that π is the theta lift of the representation

$$(\mathrm{DS}(\lambda_1-\lambda_2) \times \mathrm{DS}(\lambda_1+\lambda_2))^\sharp \tag{5.12}$$

of $H(\mathbb{R})$ to $G(\mathbb{R})$. Here $\mathrm{DS}(\kappa)$ denotes the (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with minimal weight $\kappa \in \mathbb{Z}_{\geq 1}$. Since we assume π has trivial central character, $\lambda_1 - \lambda_2$ is an even integer.

Let π be a representation of type (PS). Then π is induced from the following representation of the standard Borel subgroup

$$\begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & \nu t_1^{-1} & 0 \\ 0 & 0 & * & \nu t_2^{-1} \end{pmatrix} \longmapsto |t_1|^{\lambda_1} \cdot |t_2|^{\lambda_2} \cdot \mathrm{sgn}(\nu)^\varepsilon |\nu|^{-(\lambda_1+\lambda_2)/2} \tag{5.13}$$

for some $\varepsilon \in \{0,1\}$ and $\lambda = (\lambda_1,\lambda_2) \in \mathbb{C}^2$. Note that λ is determined by π up to the action of the Weyl group. We call ε and λ the sign and the parameter of π, respectively. Note that π is the theta lift of the representation

$$\left(\mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{(\lambda_1+\lambda_2)/2}\mathrm{sgn}^\varepsilon \boxtimes |\ |^{(-\lambda_1-\lambda_2)/2}\mathrm{sgn}^\varepsilon) \times \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{(\lambda_1-\lambda_2)/2}\mathrm{sgn}^\varepsilon \boxtimes |\ |^{(-\lambda_1+\lambda_2)/2}\mathrm{sgn}^\varepsilon)\right)^\sharp \tag{5.14}$$

of $H(\mathbb{R})$ to $G(\mathbb{R})$.

Let $\mathcal{D}$ be the domain defined by

$$\mathcal{D} = \begin{cases} \{\lambda \in \mathbb{C} \mid |\mathrm{Re}(\lambda)| < 1/2\} & \text{in Case (IIa)}, \\ \left\{\lambda = (\lambda_1,\lambda_2) \in \mathbb{C}^2 \mid |\mathrm{Re}(\lambda_1)| + |\mathrm{Re}(\lambda_2)| < 1\right\} & \text{in Case (PS)}. \end{cases} \tag{5.15}$$

By the unitarizability criterion [LMT04, Theorem 1.1], the domain $\mathcal{D}$ contains the set of parameters of unitary representations of type (IIa) or (PS).

Let π be a representation of $G(F)$ in one of the three types. When π is of type (IIa) or (PS), we assume its parameter is in $\mathcal{D}$. Let W_π be the Whittaker function of π with respect to ψ_U defined as follows:

- In Case (IIa), W_π is $\mathrm{K}(\varpi)$-invariant and $W_\pi(\mathrm{diag}(\varpi^{-\mathfrak{c}}, 1, \varpi^{2\mathfrak{c}}, \varpi^{\mathfrak{c}})) = 1$.
- In Case (DS), W_π is a lowest weight vector of the minimal $\mathrm{U}(2)$-type of $D_{(-\lambda_2,-\lambda_1)}$ in the Whittaker model of π with respect to ψ_U, and $W_\pi(1)$ is normalized as in (1.2). We refer to § 2.4 for the choice of the lowest weight vector.
- In Case (PS), W_π is $(\mathrm{Sp}_4(\mathbb{R}) \cap \mathrm{O}(4))$-invariant and $W_\pi(1)$ is normalized as in (1.3).

Since π has trivial central character, we have $\pi \simeq \pi^\vee$. Therefore $W_\pi = W_{\pi^\vee}$ when π is of type (IIa) or (PS).

When π is of type (IIa) or (PS), let ϕ_π be a matrix coefficient of π defined by

$$\phi_\pi(g) = \frac{\langle \pi(g)W_\pi, W_{\pi^\vee}\rangle}{\langle W_\pi, W_{\pi^\vee}\rangle}.$$

Here $\langle\ ,\ \rangle$ is an invariant bilinear pairing on $\pi \times \pi^\vee$. Note that if π is unitary, then

$$\phi_\pi(g) = \frac{\langle \pi(g)W_\pi, W_\pi\rangle_h}{\langle W_\pi, W_\pi\rangle_h}$$

for any Hermitian pairing $\langle\ ,\ \rangle_h$ on $\pi \times \pi$, and

$$\overline{W_\pi(g)} = W_\pi(\mathrm{diag}(-1,1,1,-1)g)$$

for all $g \in G(F)$.

When π is of type (DS), let ϕ_π be the matrix coefficient of π defined by

$$\phi_\pi(g) = \frac{\langle \pi(g)W_\pi, W_\pi \rangle}{\langle W_\pi, W_\pi \rangle}$$

for any Hermitian pairing $\langle\ ,\ \rangle$ on $\pi \times \pi$.

Let $I(s) = I_{4,4}(s)$ and $\mathcal{I}(s) = I_{4,3}(s)$ be the degenerate principal series representations of $\mathrm{GSp}_8(F)$ defined as in § 3.1. If $F \in I(s)$ and $\mathcal{F} \in \mathcal{I}(s)$ are holomorphic sections, let $Z(s, \phi_\pi, F)$ and $\mathcal{Z}(s, W_\pi, \mathcal{F})$ be the local zeta integals defined as in (4.4) and (4.5), respectively. Note that when π is of type (IIa) or (PS), we replace $\overline{W_\pi}$ by the left translation of W_π by $\mathrm{diag}(-1, 1, 1, -1)$. Recall the intertwining maps

$$S(\mathrm{M}_{6,4}(F)) \longrightarrow \mathcal{I}(s), \quad \varphi \longmapsto \mathcal{F}(\varphi),$$
$$S(\mathrm{M}_{6,4}(F)) \longrightarrow I(s), \quad \varphi \longmapsto F(\varphi),$$

defined in § 3.2 and § 3.3, respectively. By Lemma 5.3, there exists a Schwartz function $\varphi \in S(\mathrm{M}_{6,4}(F))$ such that

$$Z\left(\frac{1}{2}, \phi_\pi, F(\varphi)\right) \neq 0,$$

and such that φ does not depend on π when π is of type (IIa) or (PS). We fix one such φ and define constants $C(\pi)$ and $C'(\pi)$ as in (5.5) and (5.9), respectively.

Corollary 5.7. *Assume π is of type (IIa) or (PS) with parameter λ in $\mathcal{D}$. The constants $C(\pi)$ and $C'(\pi)$ as functions of λ are analytic.*

Proof. It is easy to show that the maps $\lambda \mapsto \phi_\pi$ and $\lambda \mapsto W_\pi$ define K-finite analytic families of matrix coefficiens and Whittaker functions in the sense introduced in § 9.1 and § 9.2 below, respectively. The assertion then follows directly from Lemmas 9.3 and 9.5. □

Proposition 5.8. *Let π be a representation in one of the three types. When π is of type (IIa) or (PS), we assume its parameter is in $\mathcal{D}$. We have*

$$C(\pi) = C'(\pi). \tag{5.16}$$

It is clear that Theorem 2.1 follows from Propositions 5.4 and 5.8. We divide the proof of Proposition 5.8 into four steps as follows:

Step 1. Establish (5.16) in Case (DS) when $\lambda_1 - \lambda_2$ and $\lambda_1 + \lambda_2$ are sufficiently large.
Step 2. Establish (5.16) in Case (IIa) when $F = \mathbb{Q}_p$.
Step 3. Establish (5.16) in Case (DS) for arbitrary parameter λ.
Step 4. Establish (5.16) in Case (PS) and in Case (IIa) for arbitrary non-archimedean F.

The ingredients are Corollary 5.6, explicit local theta correspondence (5.11), (5.12), (5.14), and the limit multiplicity formula [Shi12, §5].

5.3. **Proof of Proposition 5.8.** We need the following result on the existence of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_k)$ for a totally real number field $k \neq \mathbb{Q}$. The proposition is a simple variant of the results in [Shi12] and is well-known, but we give a proof for completeness. For simplicity, we assume $[k : \mathbb{Q}]$ is odd. In fact, we use the following result only for totally real cubic number fields.

Proposition 5.9. *Assume $k \neq \mathbb{Q}$ is a totally real number field of odd degree. Let v_0 be a real place of k and χ a unitary character of $\mathbb{R}^\times$. For each open neighborhood U of 0 in $\mathbb{R}$, there exist infinitely many irreducible cuspidal automorphic representations σ of $\mathrm{GL}_2(\mathbb{A}_k)$ with trivial central character satisfying the following conditions:*

- *σ_v is unramified for all finite places v.*
- *σ_v is a discrete series representation for all real places $v \neq v_0$.*
- *$\sigma_{v_0} = \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi|\ |^{\sqrt{-1}t} \boxtimes \chi^{-1}|\ |^{-\sqrt{-1}t})$ for some $t \in U$.*

Proof. Let S_0 be the set of real places $v \neq v_0$. Since $[k : \mathbb{Q}]$ is odd and $k \neq \mathbb{Q}$, there exists a unique quaternion division algebra D over k which is ramified precisely at the places in S_0. We write $G = D^\times/k^\times$ only in this proof and regard it as an algebraic group over k. Note that G is anisotropic over k and $G(k_v)$ is compact for all $v \in S_0$. For each finite place v, we denote by $K_v = \mathrm{PGL}_2(\mathfrak{o}_v)$ the standard maximal compact subgroup of $G(k_v) = \mathrm{PGL}_2(k_v)$, and put $K_f = \prod_{v\nmid\infty} K_v$. For each $v \in S_0$, fix a sequence $\{\tau_{v,n}\}_{n\geq 1}$ of irreducible representations of $G(k_v)$, and put $\tau_n = \bigotimes_{v\in S_0} \tau_{v,n}$. We assume that

$$\lim_{n\to\infty} \dim \tau_n = \infty.$$

We denote by $\widehat{G(k_{v_0})}$ the unitary dual of $G(k_{v_0})$ equipped with the Fell topology and by μ_{pl} the Plancherel measure on $\widehat{G(k_{v_0})}$. We define a subset $\mathcal{U}$ of $\widehat{G(k_{v_0})}$ by

$$\mathcal{U} = \left\{ \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi|\,|^{\sqrt{-1}t} \boxtimes \chi^{-1}|\,|^{-\sqrt{-1}t}) \;\middle|\; t \in U \right\}.$$

Then, by the Jacquet-Langlands correspondence, it suffices to show that for any sufficiently large n, there exists an irreducible automorphic representation π of $G(\mathbb{A}_k)$ satisfying the following conditions:

- π_v has a non-zero K_v-invariant vector for all finite places v.
- $\pi_v = \tau_{v,n}$ for all $v \in S_0$.
- $\pi_{v_0} \in \mathcal{U}$.

We now consider a sequence $\{\mu_n\}_{n\geq 1}$ of positive Borel measures on $\widehat{G(k_{v_0})}$ given by

$$\mu_n = \frac{1}{\mathrm{vol}(G(k)\backslash G(\mathbb{A}_k)) \dim \tau_n} \sum_{\rho} m_n(\rho)\delta_\rho,$$

where ρ runs over irreducible unitary representations of $G(k_{v_0})$, $m_n(\rho)$ is the multiplicity of $\rho \otimes \tau_n$ in the space of K_f-invariant automorphic forms on $G(\mathbb{A}_k)$, and δ_ρ is the Dirac measure at ρ. Then we are reduced to showing that

$$\lim_{n\to\infty} \mu_n(\mathbb{I}_{\mathcal{U}}) = \mu_{\mathrm{pl}}(\mathbb{I}_{\mathcal{U}}).$$

By the density theorem of Sauvageot [Sau97, Theorem 7.3], we are further reduced to showing that

$$\lim_{n\to\infty} \mu_n(\widehat{\phi}) = \mu_{\mathrm{pl}}(\widehat{\phi})$$

for all $\phi \in C_c^\infty(G(k_{v_0}))$, where $\widehat{\phi}(\rho) = \mathrm{tr}(\rho(\phi))$ for $\rho \in \widehat{G(k_{v_0})}$. To prove this, we use the trace formula

$$I_{\mathrm{geom}}(f) = I_{\mathrm{spec}}(f).$$

Here, for a test function $f \in C_c^\infty(G(\mathbb{A}_k))$, the geometric side $I_{\mathrm{geom}}(f)$ is given by

$$I_{\mathrm{geom}}(f) = \sum_{\gamma} \mathrm{vol}(G_\gamma(k)\backslash G_\gamma(\mathbb{A}_k)) O_\gamma(f),$$

where γ runs over conjugacy classes of $G(k)$, G_γ is the centralizer of γ in G, and $O_\gamma(f)$ is the orbital integral of f at γ. Also, the spectral side $I_{\mathrm{spec}}(f)$ is given by

$$I_{\mathrm{spec}}(f) = \sum_{\pi} m(\pi)\,\mathrm{tr}(\pi(f)),$$

where π runs over irreducible unitary representations of $G(\mathbb{A}_k)$ and $m(\pi)$ is the multiplicity of π in $L^2(G(k)\backslash G(\mathbb{A}_k))$. If we take a test function

$$f_n = \frac{1}{\mathrm{vol}(G(k)\backslash G(\mathbb{A}_k)) \dim \tau_n} \cdot \left(\bigotimes_{v\in S_0} f_{v,n} \right) \otimes \left(\bigotimes_{v\notin S_0} f_v \right)$$

given by

- $f_v = \mathbb{I}_{K_v}$ for all finite places v,
- $f_{v,n}(g) = \mathrm{tr}(\tau_{v,n}(g^{-1}))$ for all $v \in S_0$,
- $f_{v_0} = \phi$,

then

$$I_{\mathrm{spec}}(f_n) = \frac{1}{\mathrm{vol}(G(k)\backslash G(\mathbb{A}_k)) \dim \tau_n} \sum_{\rho} m_n(\rho)\,\mathrm{tr}(\rho(\phi)) = \mu_n(\widehat{\phi}).$$

On the other hand, we have

$$O_\gamma(f_n) = \frac{1}{\mathrm{vol}(G(k)\backslash G(\mathbb{A}_k))} \cdot \prod_{v\in S_0} \frac{\mathrm{vol}(G_{\gamma,v}\backslash G_v)\,\mathrm{tr}(\tau_{v,n}(\gamma^{-1}))}{\dim \tau_{v,n}} \cdot \prod_{v\notin S_0} O_\gamma(f_v).$$

If $\gamma \neq 1$, then by [CC09, Corollaire 1.12], we have

$$\lim_{n\to\infty}\prod_{v\in S_0}\frac{\mathrm{tr}(\tau_{v,n}(\gamma^{-1}))}{\dim\tau_{v,n}}=0.$$

Hence, noting that the sum in $I_{\mathrm{geom}}(f_n)$ can be taken over a finite set independent of n, we have

$$\lim_{n\to\infty} I_{\mathrm{geom}}(f_n)=\lim_{n\to\infty}\mathrm{vol}(G(k)\backslash G(\mathbb{A}_k))f_n(1)=\phi(1)=\mu_{\mathrm{pl}}(\widehat{\phi}).$$

This implies the assertion. □

Now we begin the proof of Proposition 5.8. We will use a global-to-local argument repeatedly. When k is a totally real number field, and σ_1, σ_2 are irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_k)$ with central characters inverse to each other, we put

$$(\sigma_1\times\sigma_2)^\sharp=\bigotimes_v(\sigma_{1,v}\times\sigma_{2,v})^\sharp.$$

By [Tak09, Proposition 5.4], $(\sigma_1\times\sigma_2)^\sharp$ is an irreducible cuspidal automorphic representation of $H(\mathbb{A}_k)$. Let $\theta((\sigma_1\times\sigma_2)^\sharp)$ be the global theta lift of $(\sigma_1\times\sigma_2)^\sharp$ to $G(\mathbb{A}_k)$. If $\sigma_1\not\simeq\sigma_2^\vee$, then $\theta((\sigma_1\times\sigma_2)^\sharp)$ is an irreducible unitary endoscopic cuspidal automorphic representation of $G(\mathbb{A}_k)$.

Step 1. Assume π is of type (DS) such that $\lambda_1+\lambda_2$ and $\lambda_1-\lambda_2$ are sufficiently large positive even integers. By the dimension formulae for the space of holomorphic elliptic cusp forms of full level, there exist irreducible cuspidal automorphic representations σ_1 and σ_2 of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character satisfying the following conditions:

- $\sigma_{1,p}$ and $\sigma_{2,p}$ are unramified for all primes p.
- $\sigma_{1,\infty}=\mathrm{DS}(\lambda_1-\lambda_2)$ and $\sigma_{2,\infty}=\mathrm{DS}(\lambda_1+\lambda_2)$.

Then $\theta((\sigma_1\times\sigma_2)^\sharp)$ is an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with trivial central character satisfying the following conditions:

- $\theta((\sigma_1\times\sigma_2)^\sharp)_p$ is unramified for all primes p.
- $\theta((\sigma_1\times\sigma_2)^\sharp)_\infty=\pi$.

Therefore, (5.16) follows from Corollary 5.6.

Step 2. Assume π is of type (IIa) with $F=\mathbb{Q}_p$ for some prime p. By Corollary 5.7 and the identity theorem for holomorphic functions, we may assume π is tempered, that is, $\mathrm{Re}(\lambda)=0$. Fix an irreducible cuspidal automorphic representation σ_1 of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character such that

- $\sigma_{1,\ell}$ is unramified for all primes $\ell\neq p$.
- $\sigma_{1,p}=\mathrm{St}\otimes\eta^\varepsilon$.
- $\sigma_{1,\infty}=\mathrm{DS}(\kappa_1)$ for some sufficiently large $\kappa_1\in\mathbb{Z}_{\geq 2}$.

Let U be an open neighborhood of 0 in $\mathbb{R}$. By [Shi12, Theorem 5.8], there exist infinitely many irreducible cuspidal automorphic representations σ_2 of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character such that

- $\sigma_{2,\ell}$ is unramified for all pimes $\ell\neq p$.
- $\sigma_{2,p}=\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(|\ |^{\lambda+\sqrt{-1}t}\eta^\varepsilon\boxtimes|\ |^{-\lambda-\sqrt{-1}t}\eta^\varepsilon)$ for some $t\in U$.
- $\sigma_{2,\infty}$ is a discrete series representation.

We fix one such representation σ_2 such that $\sigma_{2,\infty}$ has sufficiently large weight $\kappa_2\in\mathbb{Z}_{\geq 2}$. Then $\theta((\sigma_1\times\sigma_2)^\sharp)$ is an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with trivial central character satisfying the following conditions:

- $\theta((\sigma_1\times\sigma_2)^\sharp)_\ell$ is unramified for all primes $\ell\neq p$.
- $\theta((\sigma_1\times\sigma_2)^\sharp)_p$ is of type (IIa) with parameter $\lambda+t\sqrt{-1}$ for some $t\in U$ and sign ε.
- $\theta((\sigma_1\times\sigma_2)^\sharp)_\infty$ is of type (DS) with parameter $((\kappa_1+\kappa_2)/2,-|\kappa_1-\kappa_2|/2)$.

By Corollary 5.6,

$$C(\theta((\sigma_1\times\sigma_2)^\sharp)_p)C(\theta((\sigma_1\times\sigma_2)^\sharp)_\infty)=C'(\theta((\sigma_1\times\sigma_2)^\sharp)_p)C'(\theta((\sigma_1\times\sigma_2)^\sharp)_\infty).$$

On the other hand, since κ_1 and κ_2 are sufficiently large,

$$C(\theta((\sigma_1\times\sigma_2)^\sharp)_\infty)=C'(\theta((\sigma_1\times\sigma_2)^\sharp)_\infty)$$

by **Step 1**. We conclude that

$$C(\theta((\sigma_1 \times \sigma_2)^\sharp)_p) = C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_p).$$

Therefore, (5.16) follows from Corollary 5.7.

Step 3. Assume π is of type (DS) for arbitrary parameter λ. Recall that $\lambda_1 + \lambda_2$ and $\lambda_1 - \lambda_2$ are positive even integers. By the dimension formulae for the space of holomorphic elliptic cusp forms, there exist two distinct primes p_1, p_2, and irreducible cuspidal automorphic representations σ_1 and σ_2 of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with trivial central character such that

- $\sigma_{1,\ell}$ is unramified for all primes $\ell \neq p_1$.
- $\sigma_{2,\ell}$ is unramified for all primes $\ell \neq p_2$.
- σ_{1,p_1} and σ_{2,p_2} are special representations of conductor $p_1\mathbb{Z}_{p_1}$ and $p_2\mathbb{Z}_{p_2}$, respectively.
- $\sigma_{1,\infty} = \mathrm{DS}(\lambda_1 - \lambda_2)$ and $\sigma_{2,\infty} = \mathrm{DS}(\lambda_1 + \lambda_2)$ are discrete series representations.

Then $\theta((\sigma_1 \times \sigma_2)^\sharp)$ is an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ with trivial central character satisfying the following conditions:

- $\theta((\sigma_1 \times \sigma_2)^\sharp)_\ell$ is unramified for all primes $\ell \notin \{p_1, p_2\}$.
- $\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_i}$ is of type (IIa) for $i = 1, 2$.
- $\theta((\sigma_1 \times \sigma_2)^\sharp)_\infty = \pi$.

By Corollary 5.6,

$$C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_1})C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_2})C(\pi) = C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_1})C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_2})C'(\pi).$$

On the other hand, by **Step 2**,

$$C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_1})C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_2}) = C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_1})C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{p_2}).$$

Therefore (5.16) holds for π.

Step 4. First we assume π is of type (IIa). There exist a totally real number field k and a finite place v_0 of k such that $k_{v_0} = F$. To prove (5.16), we proceed as in **Step 2**. By Corollary 5.7 and the identity theorem for holomorphic functions, we may assume $\mathrm{Re}(\lambda) = 0$. By [Shi12, Theorem 5.8] (see also [Wei09, Theorem 1.1]), there exists an irreducible cuspidal automorphic representation σ_1 of $\mathrm{GL}_2(\mathbb{A}_k)$ with trivial central character satisfying the following conditions:

- $\sigma_{1,v}$ is unramified for all finite places $v \neq v_0$.
- $\sigma_{1,v_0} = \mathrm{St} \otimes \eta^\varepsilon$.
- $\sigma_{1,v}$ is a discrete series representation for all real places v.

Let U be an open neighborhood of 0 in $\mathbb{R}$. By [Shi12, Theorem 5.8], there exists an irreducible cuspidal automorphic representation σ_2 of $\mathrm{GL}_2(\mathbb{A}_k)$ with trivial central character such that

- $\sigma_{2,v}$ is unramified for all finite places $v \neq v_0$.
- $\sigma_{2,v_0} = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(|\ |^{\lambda+\sqrt{-1}t}\eta^\varepsilon \boxtimes |\ |^{-\lambda-\sqrt{-1}t}\eta^\varepsilon)$ for some $t \in U$.
- $\sigma_{2,v}$ is a discrete series representation for all real places v.

Then $\theta((\sigma_1 \times \sigma_2)^\sharp)$ is an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A}_k)$ with trivial central character satisfying the following conditions:

- $\theta((\sigma_1 \times \sigma_2)^\sharp)_v$ is unramified for all finite places $v \neq v_0$.
- $\theta((\sigma_1 \times \sigma_2)^\sharp)_{v_0}$ is of type (IIa) with parameter $\lambda + t\sqrt{-1}$ for some $t \in U$ and sign ε.
- $\theta((\sigma_1 \times \sigma_2)^\sharp)_v$ is of type (DS) for all real places v.

By Corollary 5.6,

$$C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{v_0}) \prod_{v|\infty} C(\theta((\sigma_1 \times \sigma_2)^\sharp)_v) = C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{v_0}) \prod_{v|\infty} C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_v).$$

On the other hand, by **Step 3**,

$$\prod_{v|\infty} C(\theta((\sigma_1 \times \sigma_2)^\sharp)_v) = \prod_{v|\infty} C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_v).$$

We conclude that

$$C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{v_0}) = C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{v_0}).$$

Therefore, (5.16) follows from Corollary 5.7.

Finally we assume π is of type (PS). By Corollary 5.7 and the identity theorem for holomorphic functions, we may assume π is tempered, that is, $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = 0$. Let k be a totally real cubic number field. Let $\{\infty_1, \infty_2, \infty_3\}$ be the set of real places of k. Let U be an open neighborhood of 0 in $\mathbb{R}$. By Proposition 5.9, there exist irreducible cuspidal automorphic representations σ_1 and σ_2 of $\mathrm{GL}_2(\mathbb{A}_k)$ with trivial central character satisfying the following conditions:

- $\sigma_{i,v}$ is unramified for all finite places v and $i = 1, 2$.
- σ_{i,∞_1} and σ_{i,∞_2} are discrete series representations for $i = 1, 2$.
- $\sigma_{1,\infty_3} = \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{(\lambda_1+\lambda_2)/2+\sqrt{-1}t_1}\mathrm{sgn}^{\varepsilon} \boxtimes |\ |^{(-\lambda_1-\lambda_2)/2-\sqrt{-1}t_1}\mathrm{sgn}^{\varepsilon})$ for some $t_1 \in U$.
- $\sigma_{2,\infty_3} = \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{(\lambda_1-\lambda_2)/2+\sqrt{-1}t_2}\mathrm{sgn}^{\varepsilon} \boxtimes |\ |^{(-\lambda_1+\lambda_2)/2-\sqrt{-1}t_2}\mathrm{sgn}^{\varepsilon})$ for some $t_2 \in U$.

Then $\theta((\sigma_1 \times \sigma_2)^\sharp)$ is an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A}_k)$ with trivial central character satisfying the following conditions:

- $\theta((\sigma_1 \times \sigma_2)^\sharp)_v$ is unramified for all finite places v.
- $\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_i}$ is of type (DS) for $i = 1, 2$.
- $\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_3}$ is of type (PS) with parameter $(\lambda_1 + \sqrt{-1}(t_1 + t_2), \lambda_2 + \sqrt{-1}(t_1 - t_2))$ and sign ε.

By Corollary 5.6,

$$\prod_{i=1}^{3} C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_i}) = \prod_{i=1}^{3} C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_i}).$$

On the other hand, by **Step 3**,

$$C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_1})C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_2}) = C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_1})C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_2}).$$

We conclude that

$$C(\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_3}) = C'(\theta((\sigma_1 \times \sigma_2)^\sharp)_{\infty_3}).$$

Therefore, (5.16) follows from Corollary 5.7. This completes the proof of Proposition 5.8.

6. Endoscopic lifts

Let π be an irreducible globally generic cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character and paramodular conductor $\mathfrak{n}$ satisfying the conditions in § 2.4. In this section, we assume π is endoscopic. Then we state the key ingredients (Propositions 6.1 and 6.2) of the explicit Rallis inner product formula (Proposition 5.5).

6.1. **Theta lifts.** Let (V, Q) be the quadratic space over k defined by $V = \mathrm{M}_{2,2}$ and $Q[x] = \det(x)$. Let ι be the main involution on $\mathrm{M}_{2,2}$ defined by

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^{\iota} = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}.$$

The associated symmetric bilinear form is given by $(x, y) = \mathrm{tr}(xy^\iota)$. Put

$$H = \mathrm{GO}(V), \quad H_1 = \mathrm{O}(V),$$
$$H^\circ = \mathrm{GSO}(V), \quad H_1^\circ = \mathrm{SO}(V).$$

Let $\nu : H \to \mathbb{G}_m$ be the similitude character. Let $\mathbf{t}$ be the involution on $\mathrm{GL}_2 \times \mathrm{GL}_2$ defined by

$$\mathrm{Ad}(\mathbf{t})(h_1, h_2) = \left((h_2^\iota)^{-1}, (h_1^\iota)^{-1}\right).$$

Put $\boldsymbol{\mu}_2 = \langle \mathbf{t} \rangle$. We have an exact sequence

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\Delta} (\mathrm{GL}_2 \times \mathrm{GL}_2) \rtimes \boldsymbol{\mu}_2 \xrightarrow{\rho} H \longrightarrow 1,$$

where $\Delta(a) = (a\mathbf{1}_2, a\mathbf{1}_2)$, $\rho(h_1, h_2)x = h_1 x h_2^{-1}$, and $\rho(\mathbf{t})x = x^\iota$ for $a \in \mathbb{G}_m, h_1, h_2 \in \mathrm{GL}_2$, and $x \in V$. For $h_1, h_2 \in \mathrm{GL}_2$, we write $\rho(h_1, h_2) = [h_1, h_2]$. Note that $\nu([h_1, h_2]) = \det(h_1 h_2^{-1})$. We denote by $\mathbf{t}$ the image of $\mathbf{t}$ in H by abuse of notation, and by $\mathbf{t}_v$ the image of $\mathbf{t}$ in $H(k_v)$ for each place v of k.

Let $\omega = \omega_{\psi,V,2}$ be the Weil representation of $\mathrm{Sp}_4(\mathbb{A}) \times H_1(\mathbb{A})$ on $\mathcal{S}(V^2(\mathbb{A}))$ with respect to ψ. Let $S(V^2(\mathbb{A}))$ be the subspace of $\mathcal{S}(V^2(\mathbb{A}))$ consisting of functions which correspond to polynomials in the Fock model at the archimedean places. We extend ω to a representation of $\mathrm{G}(\mathrm{Sp}_4 \times H_1)(\mathbb{A})$ as in (3.1).

Let $\varphi \in S(V^2(\mathbb{A}))$. The theta function associated to φ is defined by

$$\Theta(g, h; \varphi) = \sum_{x \in V^2(k)} \omega(g, h)\varphi(x)$$

for $(g, h) \in \mathrm{G}(\mathrm{Sp}_4 \times H_1)(\mathbb{A})$. Let f be a cusp form on $H(\mathbb{A})$ and let $\varphi \in S(V^2(\mathbb{A}))$. For $g \in G(\mathbb{A})$, choose $h \in H(\mathbb{A})$ such that $\nu(h) = \nu(g)$, and put

$$\theta(f, \varphi)(g) = \int_{H_1(k)\backslash H_1(\mathbb{A})} f(h_1 h)\Theta(g, h_1 h; \varphi)\, dh_1.$$

Then $\theta(f, \varphi)$ is an automorphic form on $G(\mathbb{A})$. For any irreducible cuspidal automorphic representation $\sigma^\sharp$ of $H(\mathbb{A})$ on the space $\mathcal{V}_{\sigma^\sharp}$, we define $\theta(\sigma^\sharp)$ as the automorphic representation of $G(\mathbb{A})$ on the space $\mathcal{V}_{\theta(\sigma^\sharp)}$ generated by $\theta(f, \varphi)$ for all $f \in \mathcal{V}_{\sigma^\sharp}$ and $\varphi \in S(V^2(\mathbb{A}))$.

Since we assume π is endoscopic, by [AS06, Proposition 2.2], there exists an irreducible unitary cuspidal automorphic representation $\sigma^\sharp$ of $H(\mathbb{A})$ such that

$$\theta(\sigma^\sharp) = \pi.$$

By [HST93, Lemma 2], the tower property, and the cuspidality of π, we have

$$\mathcal{V}_{\sigma^\sharp}|_{H^\circ(\mathbb{A})} = \mathcal{V}_\sigma \oplus \mathcal{V}_{\sigma \circ \mathrm{Ad}(\mathbf{t})} \tag{6.1}$$

as spaces of functions on $H^\circ(\mathbb{A})$ for some irreducible unitary cuspidal automorphic representation σ of $H^\circ(\mathbb{A})$ on the space $\mathcal{V}_\sigma$ with $\sigma \neq \sigma \circ \mathrm{Ad}(\mathbf{t})$. Via the isomorphism

$$H^\circ \simeq \Delta\mathbb{G}_m\backslash(\mathrm{GL}_2 \times \mathrm{GL}_2)$$

induced by ρ, we have

$$\sigma = \sigma_1 \times \sigma_2, \quad \mathcal{V}_\sigma = \mathcal{V}_{\sigma_1} \otimes \mathcal{V}_{\sigma_2}$$

for some irreducible unitary cuspidal automorphic representations σ_1 and σ_2 of $\mathrm{GL}_2(\mathbb{A})$ on the spaces $\mathcal{V}_{\sigma_1}$ and $\mathcal{V}_{\sigma_2}$, respectively, with central characters inverse to each other. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be the conductors of σ_1 and σ_2, respectively. By our assumptions on π in § 2.4 and the condition $\sigma \neq \sigma \circ \mathrm{Ad}(\mathbf{t})$, σ_1 and σ_2 satisfy the following properties:

- $\sigma_1 \neq \sigma_2^\vee$.
- σ_1 and σ_2 have trivial central characters.
- $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are square-free and coprime, and $\mathfrak{n}_1\mathfrak{n}_2 = \mathfrak{n}$.
- For $v \in S(\mathrm{DS})$, $\sigma_{1,v}$ and $\sigma_{2,v}$ are discrete series representations.
- For $v \in S(\mathrm{PS})$, $\sigma_{1,v}$ and $\sigma_{2,v}$ are unitary principal series representations with non-zero $\mathrm{SO}(2)$-invariant vectors.

For $v \in S(\mathrm{DS})$, let $\kappa_{1,v} \in \mathbb{Z}_{\geq 1}$ and $\kappa_{2,v} \in \mathbb{Z}_{\geq 1}$ be the minimal weights of $\sigma_{1,v}$ and $\sigma_{2,v}$, respectively. Then

$$\lambda_{1,v} = \frac{\kappa_{1,v} + \kappa_{2,v}}{2}, \quad \lambda_{2,v} = -\frac{|\kappa_{1,v} - \kappa_{2,v}|}{2}.$$

For $v \in S(\mathrm{PS})$, we have

$$\sigma_{i,v} = \mathrm{Ind}_{B(k_v)}^{\mathrm{GL}_2(k_v)}(|\ |_v^{\mu_{i,v}}\mathrm{sgn}^\varepsilon \boxtimes |\ |_v^{-\mu_{i,v}}\mathrm{sgn}^\varepsilon)$$

for some $\varepsilon \in \{0, 1\}$ and $\mu_{i,v} \in \mathbb{C}$ with $|\mathrm{Re}(\mu_{i,v})| < 1/2$ for $i = 1, 2$. Then

$$\{\lambda_{1,v}, \lambda_{2,v}\} = \{\mu_{1,v} + \mu_{2,v}, \mu_{1,v} - \mu_{2,v}\},$$

after replacing $\lambda_{i,v}$ with $-\lambda_{i,v}$ or $\mu_{i,v}$ with $-\mu_{i,v}$ if necessary.

6.2. **Automorphic forms on** $\mathrm{GO}(V)$**.** Consider non-zero automorphic forms $\mathbf{f}_1 \in \mathcal{V}_{\sigma_1}$ and $\mathbf{f}_2 \in \mathcal{V}_{\sigma_2}$ satisfying the following conditions:

- If v is a finite place and $v \nmid \mathfrak{n}$, then
$$\sigma_{1,v}(k_1)\mathbf{f}_1 = \mathbf{f}_1, \quad \sigma_{2,v}(k_2)\mathbf{f}_2 = \mathbf{f}_2$$
for $(k_1, k_2) \in \mathrm{GL}_2(\mathfrak{o}_v) \times \mathrm{GL}_2(\mathfrak{o}_v)$.
- If $v \mid \mathfrak{n}_1$, then
$$\sigma_{1,v}(k_1)\mathbf{f}_1 = \mathbf{f}_1, \quad \sigma_{2,v}(k_2)\mathbf{f}_2 = \mathbf{f}_2$$
for $(k_1, k_2) \in K_0(\varpi_v) \times \mathrm{GL}_2(\mathfrak{o}_v)$.

- If $v \mid \mathfrak{n}_2$, then
$$\sigma_{1,v}(k_1)\mathbf{f}_1 = \mathbf{f}_1, \quad \sigma_{2,v}(k_2)\mathbf{f}_2 = \mathbf{f}_2$$
for $(k_1, k_2) \in \mathrm{GL}_2(\mathfrak{o}_v) \times K_0(\varpi_v)$.
- If $v \in S(\mathrm{DS})$, then
$$\sigma_{1,v}(k_\theta)\mathbf{f}_1 = e^{\sqrt{-1}\kappa_{1,v}\theta}\mathbf{f}_1, \quad \sigma_{2,v}(k_\theta)\mathbf{f}_2 = e^{\sqrt{-1}\kappa_{2,v}\theta}\mathbf{f}_2$$
for $k_\theta \in \mathrm{SO}(2)$.
- If $v \in S(\mathrm{PS})$, then
$$\sigma_{1,v}(k)\mathbf{f}_1 = \mathbf{f}_1, \quad \sigma_{2,v}(k)\mathbf{f}_2 = \mathbf{f}_2$$
for $k \in \mathrm{SO}(2)$.

The conditions above characterize $\mathbf{f}_1$ and $\mathbf{f}_2$ up to scalars. We normalize $\mathbf{f}_1$ and $\mathbf{f}_2$ so that
$$W_{\mathbf{f}_i}\left(\prod_{v\nmid\infty}\mathbf{a}(\varpi_v^{-\mathfrak{c}_v})\right) = e^{-2\pi|S(\mathrm{DS})|}\prod_{v\in S(\mathrm{PS})} K_{\mu_{i,v}}(2\pi),$$
where $W_{\mathbf{f}_i}$ is the Whittaker function of $\mathbf{f}_i$ defined by
$$W_{\mathbf{f}_i}(g) = \int_{k\backslash\mathbb{A}} \mathbf{f}_i(\mathbf{n}(x)g)\overline{\psi(x)}\,dx.$$
Here dx is the Tamagawa measure on $\mathbb{A}$. Let $\mathbf{f} \in \mathcal{V}_\sigma$ be the automorphic form defined by
$$\mathbf{f}(h) = \mathbf{f}_1(h_1)\mathbf{f}_2(h_2)$$
for $h = [h_1, h_2] \in H^\circ(\mathbb{A})$.

Consider a non-zero automorphic form $\mathbf{f}^\sharp \in \mathcal{V}_{\sigma^\sharp}$ satisfying the following conditions:
- If v is a finite place and $v \nmid \mathfrak{n}$, then
$$\sigma_v^\sharp(k')\mathbf{f}^\sharp = \mathbf{f}^\sharp$$
for $k' \in H(\mathfrak{o}_v)$.
- If $v \mid \mathfrak{n}_1$, then
$$\sigma_v^\sharp([k_1, k_2])\mathbf{f}^\sharp = \mathbf{f}^\sharp$$
for $(k_1, k_2) \in K_0(\varpi_v) \times \mathrm{GL}_2(\mathfrak{o}_v)$.
- If $v \mid \mathfrak{n}_2$, then
$$\sigma_v^\sharp([k_1, k_2])\mathbf{f}^\sharp = \mathbf{f}^\sharp$$
for $(k_1, k_2) \in \mathrm{GL}_2(\mathfrak{o}_v) \times K_0(\varpi_v)$.
- If $v \in S(\mathrm{DS})$, then
$$\sigma_v^\sharp([k_{\theta_1}, k_{\theta_2}])\mathbf{f}^\sharp = e^{\sqrt{-1}(\kappa_{1,v}\theta_1 + \kappa_{2,v}\theta_2)}\mathbf{f}^\sharp$$
for $k_{\theta_1}, k_{\theta_2} \in \mathrm{SO}(2)$.
- If $v \in S(\mathrm{PS})$, then
$$\sigma_v^\sharp([k_1, k_2])\mathbf{f}^\sharp = \mathbf{f}^\sharp, \quad \sigma_v^\sharp(\mathbf{t}_v)\mathbf{f}^\sharp = \mathbf{f}^\sharp$$
for $k_1, k_2 \in \mathrm{SO}(2)$.

The conditions above characterize $\mathbf{f}^\sharp$ up to scalars. We normalize $\mathbf{f}^\sharp$ so that
$$\mathbf{f}^\sharp|_{H^\circ(\mathbb{A})} = \mathbf{f}.$$

Let L be the lattice of $V^2(k)$ defined by
$$L = \begin{pmatrix} \mathfrak{n}_2 & \mathfrak{o} \\ \mathfrak{n} & \mathfrak{n}_1 \end{pmatrix} \oplus \begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{pmatrix}.$$
Define $\varphi = \bigotimes_v \varphi_v \in S(V^2(\mathbb{A}))$ as follows: If v is a finite place, then

(6.2) $$\varphi_v = \mathbb{I}_{L\otimes_{\mathfrak{o}}\mathfrak{o}_v}.$$

If $v \in S(\mathrm{DS})$ and $\kappa_{1,v} \geq \kappa_{2,v}$, then

(6.3) $$\varphi_v(x, y) = (-\sqrt{-1}x_1 - x_2 - x_3 + \sqrt{-1}x_4)^{\lambda_{1,v}}(y_1 + \sqrt{-1}y_2 - \sqrt{-1}y_3 + y_4)^{-\lambda_{2,v}}e^{-\pi\,\mathrm{tr}(x^tx + y^ty)}.$$

If $v \in S(\mathrm{DS})$ and $\kappa_{1,v} \le \kappa_{2,v}$, then

$$\varphi_v(x,y) = (-\sqrt{-1}x_1 - x_2 - x_3 + \sqrt{-1}x_4)^{\lambda_{1,v}}(y_1 - \sqrt{-1}y_2 + \sqrt{-1}y_3 + y_4)^{-\lambda_{2,v}} e^{-\pi \operatorname{tr}(x^t x + y^t y)}. \tag{6.4}$$

If $v \in S(\mathrm{PS})$, then

$$\varphi_v(x,y) = e^{-\pi \operatorname{tr}(x^t x + y^t y)}. \tag{6.5}$$

Let $\mathfrak{S}$ be the subset of places of k defined by

$$\mathfrak{S} = \{v \mid \sigma_v \not\simeq \sigma_v \circ \mathrm{Ad}(\mathbf{t}_v)\}.$$

Since $\sigma \neq \sigma \circ \mathrm{Ad}(\mathbf{t})$, $\mathfrak{S}$ contains infinitely many places of k by the strong multiplicity one theorem for H°. Recall $S(\mathrm{DS})$ is the set of real places of type (DS). Let $S(\mathfrak{n}) = \{v \mid \mathfrak{n}\}$. Note that $S(\mathfrak{n}) \subseteq \mathfrak{S}$.

Proposition 6.1. *Let $f \in \pi$ be the cusp form satisfying the conditions (2.4) and (2.5). We have*

$$\theta(\mathbf{f}^\sharp, \varphi)(g) = 2^{-|(S(\mathrm{DS}) \cup S(\mathfrak{n})) \cap \mathfrak{S}|} \cdot \prod_{v \in S(\mathrm{DS})} 2^{-\lambda_{1,v}-4} \pi^{(-3\lambda_{1,v}+\lambda_{2,v}-5)/2} \cdot \prod_{v \mid \mathfrak{n}} (1+q_v)^{-1} \cdot \mathfrak{D}^{-3/2} \cdot \xi(2)^{-2} \cdot f(gg_0)$$

for $g \in G(\mathbb{A})$. Here $g_0 \in G(\mathbb{A})$ is defined by

$$g_{0,v} = \begin{cases} \mathrm{diag}(1,1,\varpi_v^{\mathfrak{c}_v}, \varpi_v^{\mathfrak{c}_v}) & \text{if } v \nmid \infty, \\ 1 & \text{if } v \mid \infty. \end{cases}$$

Proof. The proof will be given in § 7.4 below. □

Proposition 6.2. *We have*

$$\langle \theta(\mathbf{f}^\sharp, \varphi), \theta(\mathbf{f}^\sharp, \varphi) \rangle = 2^{-2|(S(\mathrm{DS}) \cup S(\mathfrak{n})) \cap \mathfrak{S}|} \cdot \mathfrak{D}^{-8} \cdot \frac{4}{\xi(2)^4} \cdot \frac{L(1,\pi,\mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot \prod_{v \mid \mathfrak{n}} q_v^{-3} \zeta_v(1)^{-2} \zeta_v(2) \zeta_v(4)$$
$$\times \prod_{v \in S(\mathrm{DS})} 2^{-\lambda_{1,v}-\lambda_{2,v}-3}(1+\lambda_{1,v}-\lambda_{2,v})^{-1} \cdot \prod_{v \in S(\mathrm{PS})} 2^{-4}.$$

Proof. The proof will be given in § 8.4 below. □

7. Calculation of Whittaker functions

We keep the notation of § 6. The aim of this section is to prove Proposition 6.1. We prove it by comparing the Whittaker function of $\theta(\mathbf{f}^\sharp, \varphi)$ and the normalization of f in (2.5). It boils down to the calculation of certain local Whittaker functions of π_v in terms of local integrals involving Whittaker functions of σ_v and the Weil representation ω_v for each place v.

7.1. **Measures.** Let v be a place of k. Let K_v be the maximal compact subgroup of $H_1^\circ(k_v)$ defined by

$$K_v = \begin{cases} [\mathbf{d}(\varpi_v^{\mathfrak{c}}), \mathbf{d}(\varpi_v^{\mathfrak{c}})] H_1^\circ(\mathfrak{o}_v) [\mathbf{d}(\varpi_v^{-\mathfrak{c}}), \mathbf{d}(\varpi_v^{-\mathfrak{c}})] & \text{if } v \text{ is finite}, \\ H_1^\circ(k_v) \cap (\mathrm{O}(2) \times \mathrm{O}(2))/\{\pm 1\} & \text{if } v \text{ is real}. \end{cases}$$

We fix a Haar measure $dh_{1,v}$ on $H_1^\circ(k_v)$ as follows:

- If v is finite, the measure is normalized so that

$$\mathrm{vol}(K_v, dh_{1,v}) = 1. \tag{7.1}$$

  For $\phi \in L^1(H_1^\circ(k_v))$, we have

(7.2)
$$\int_{H_1^\circ(k_v)} \phi(h_{1,v})\, dh_{1,v}$$
$$= q_v^{-2\mathfrak{c}_v} \int_{k_v} \int_{k_v} \int_{k_v^\times} \int_{k_v^\times} \int_{K_v} \phi([\mathbf{n}(x_1), \mathbf{n}(x_2)][\mathbf{m}(y_1), \mathbf{m}(y_2)]k) |y_1|_v^{-2} |y_2|_v^{-2}\, dk\, d^\times y_1\, d^\times y_2\, dx_1\, dx_2$$
$$+ q_v^{-2\mathfrak{c}_v} \int_{k_v} \int_{k_v} \int_{k_v^\times} \int_{k_v^\times} \int_{K_v} \phi([\mathbf{a}(\varpi_v), \mathbf{a}(\varpi_v)][\mathbf{n}(x_1), \mathbf{n}(x_2)][\mathbf{m}(y_1), \mathbf{m}(y_2)]k) |y_1|_v^{-2} |y_2|_v^{-2}\, dk\, d^\times y_1\, d^\times y_2\, dx_1\, dx_2.$$

- If v is real, the measure is defined so that for all $\phi \in L^1(H_1^\circ(\mathbb{R}))$, we have

(7.3)
$$\int_{H_1^\circ(\mathbb{R})} \phi(h_{1,\infty})\, dh_{1,\infty} = \int_{H_1^\circ(\mathbb{R})^0} \phi(h_{1,\infty}) dh_{1,\infty} + \int_{H_1^\circ(\mathbb{R})^0} \phi([\mathbf{a}(-1), \mathbf{a}(-1)]h_{1,\infty})\, dh_{1,\infty},$$
$$\int_{H_1^\circ(\mathbb{R})^0} \phi(h_{1,\infty})\, dh_{1,\infty} = 4 \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \int_0^\infty \int_{K_\infty^0} \phi([\mathbf{n}(x_1), \mathbf{n}(x_2)][\mathbf{m}(y_1), \mathbf{m}(y_2)]k) y_1^{-2} y_2^{-2}\, dk\, d^\times y_1\, d^\times y_2\, dx_1\, dx_2$$
$$= 16\pi^2 \int_{K_\infty^0} \int_0^\infty \int_0^\infty \int_{K_\infty^0} \phi(k[\mathbf{m}(e^{t_1}), \mathbf{m}(e^{t_2})]k') \sinh(2t_1) \sinh(2t_2)\, dk\, dt_1\, dt_2\, dk'.$$

  Here K_∞^0 and $H_1^\circ(\mathbb{R})^0$ are the topological identity components of K_∞ and $H_1^\circ(\mathbb{R})$, respectively, and $\mathrm{vol}(K_\infty^0, dk) = \mathrm{vol}(K_\infty^0, dk') = 1$. We refer to [II10, §12] for the measure with respect to the Cartan decomposition.

Let $d\epsilon_v$ be the Haar measure on $\boldsymbol{\mu}_2(k_v)$ such that $\mathrm{vol}(\boldsymbol{\mu}_2(k_v), d\epsilon_v) = 1$. We extend the measure on $H_1^\circ(k_v)$ to a measure on $H_1(k_v)$ by
$$\int_{H_1(k_v)} \phi(h_{1,v})\, dh_{1,v} = \int_{\boldsymbol{\mu}_2(k_v)} \int_{H_1^\circ(k_v)} \phi(h_{1,v}\epsilon_v)\, dh_{1,v}\, d\epsilon_v$$
for $\phi \in L^1(H_1(k_v))$.

Let dh_1 be the Tamagawa measure on $H_1(\mathbb{A})$. Note that $\mathrm{vol}(H_1(k)\backslash H_1(\mathbb{A})) = 1$ and
$$dh_1 = \mathfrak{D}^{-3} \cdot \xi(2)^{-2} \cdot \prod_v dh_{1,v}. \tag{7.4}$$
Let $d\epsilon = \prod_v d\epsilon_v$ be the Tamagawa measure on $\boldsymbol{\mu}_2(\mathbb{A})$.

7.2. **Whittaker functions.** Recall $\mathfrak{S}$ is the subset of places of k defined by
$$\mathfrak{S} = \{v \mid \sigma_v \not\simeq \sigma_v \circ \mathrm{Ad}(\mathbf{t}_v)\}.$$
Let v be a place of k. Let $\mathcal{V}_{1,v}$ and $\mathcal{V}_{2,v}$ be the spaces of $\sigma_{1,v}$ and $\sigma_{2,v}$, respectively. Then $\mathcal{V}_v = \mathcal{V}_{1,v} \otimes \mathcal{V}_{2,v}$ is the space of σ_v. We define an irreducible admissible representation $\sigma_v^\sharp$ of $H(k_v)$ as follows:

- $v \notin \mathfrak{S}$: In this case, $\sigma_{1,v} \simeq \sigma_{2,v}$. We take $\mathcal{V}_{1,v} = \mathcal{V}_{2,v}$ and define $\mathcal{V}_v^\sharp = \mathcal{V}_v$. The representation $\sigma_v^\sharp$ of $H(k_v)$ on $\mathcal{V}_v^\sharp$ is defined by
$$\sigma_v^\sharp([h_1, h_2])(v_1 \otimes v_2) = \sigma_{1,v}(h_1)v_1 \otimes \sigma_{2,v}(h_2)v_2,$$
$$\sigma_v^\sharp(\mathbf{t}_v)(v_1 \otimes v_2) = v_2 \otimes v_1$$
  for $h_1, h_2 \in \mathrm{GL}_2(k_v)$ and $v_1 \otimes v_2 \in \mathcal{V}_v^\sharp$.
- $v \in \mathfrak{S}$: In this case, define $\sigma_v^\sharp = \mathrm{Ind}_{H^\circ(k_v)}^{H(k_v)} \sigma_v$. The induced representation is realized on $\mathcal{V}_v^\sharp = \mathcal{V}_v \oplus \mathcal{V}_v$, where the action is defined by
$$\sigma_v^\sharp(h)(v_1, v_2) = (\sigma_v(h)v_1, \sigma_v(\mathrm{Ad}(\mathbf{t}_v)h)v_2),$$
$$\sigma_v^\sharp(\mathbf{t}_v)(v_1, v_2) = (v_2, v_1)$$
  for $h \in H^\circ(k_v)$ and $(v_1, v_2) \in \mathcal{V}_v^\sharp$.

7.2.1. *Local models.* Let v be place of k. For $i = 1, 2$, let $\mathcal{V}_{i,v} = \mathcal{W}(\sigma_{i,v}, \psi_v)$ be the space of Whittaker functions of $\sigma_{i,v}$ with respect to ψ_v. Consider a non-zero Whittaker function $W_{1,v} \in \mathcal{V}_{1,v}$ and $W_{2,v} \in \mathcal{V}_{2,v}$ satisfying the following conditions:

- If v is a finite place and $v \nmid \mathfrak{n}$, then
$$\sigma_{1,v}(k_1)W_{1,v} = W_{1,v}, \quad \sigma_{2,v}(k_2)W_{2,v} = W_{2,v}$$
  for $(k_1, k_2) \in \mathrm{GL}_2(\mathfrak{o}_v) \times \mathrm{GL}_2(\mathfrak{o}_v)$.
- If $v \mid \mathfrak{n}_1$, then
$$\sigma_{1,v}(k_1)W_{1,v} = W_{1,v}, \quad \sigma_{2,v}(k_2)W_{2,v} = W_{2,v}$$
  for $(k_1, k_2) \in K_0(\varpi_v) \times \mathrm{GL}_2(\mathfrak{o}_v)$.

- If $v \mid \mathfrak{n}_2$, then
$$\sigma_{1,v}(k_1)W_{1,v} = W_{1,v}, \quad \sigma_{2,v}(k_2)W_{2,v} = W_{2,v}$$
for $(k_1, k_2) \in \mathrm{GL}_2(\mathfrak{o}_v) \times K_0(\varpi_v)$.
- If $v \in S(\mathrm{DS})$, then
$$\sigma_{1,v}(k_\theta)W_{1,v} = e^{\sqrt{-1}\kappa_{1,v}\theta}W_{1,v}, \quad \sigma_{2,v}(k_\theta)W_{2,v} = e^{\sqrt{-1}\kappa_{2,v}\theta}W_{2,v}$$
for $k_\theta \in \mathrm{SO}(2)$.
- If $v \in S(\mathrm{PS})$, then
$$\sigma_{1,v}(k)W_{1,v} = W_{1,v}, \quad \sigma_{2,v}(k)W_{2,v} = W_{2,v}$$
for $k \in \mathrm{SO}(2)$.

The conditions above characterize $W_{1,v}$ and $W_{2,v}$ up to scalars. We normalize $W_{1,v}$ and $W_{2,v}$ as follows:

- If v is a finite place, then
$$W_{1,v}(\mathbf{a}(\varpi_v^{-\mathfrak{c}_v})) = W_{2,v}(\mathbf{a}(\varpi_v^{-\mathfrak{c}_v})) = 1.$$
- If $v \in S(\mathrm{DS})$, then
$$W_{1,v}(1) = W_{2,v}(1) = e^{-2\pi}.$$
- If $v \in S(\mathrm{PS})$, then
$$W_{1,v}(1) = K_{\mu_{1,v}}(2\pi), \quad W_{2,v}(1) = K_{\mu_{2,v}}(2\pi).$$

Let W_v be the Whittaker function on $H^\circ(k_v)$ defined by

$$W_v([h_1, h_2]) = W_1(h_1)W_2(h_2) \tag{7.5}$$

for $h_1, h_2 \in \mathrm{GL}_2(k_v)$.

Fix isomorphisms $\sigma \simeq \bigotimes_v \sigma_v$ and $\sigma^\sharp \simeq \bigotimes_v \sigma_v^\sharp$ such that

$$\begin{aligned} &\mathbf{f} \longmapsto \bigotimes_v W_v, \\ &\mathbf{f}^\sharp \longmapsto \left(\bigotimes_{\substack{v \in S(\mathrm{DS}) \cup S(\mathfrak{n}) \\ v \in \mathfrak{S}}} (W_v, 0) \right) \otimes \left(\bigotimes_{\substack{v \notin S(\mathrm{DS}) \cup S(\mathfrak{n}) \\ v \in \mathfrak{S}}} (W_v, W_v) \right) \otimes \left(\bigotimes_{v \notin \mathfrak{S}} W_v \right). \end{aligned} \tag{7.6}$$

7.2.2. *Global Whittaker functions.* If R is a set of places of k, denote by $\mathbf{t}_R \in \boldsymbol{\mu}_2(\mathbb{A})$ the element such that

$$\mathbf{t}_{R,v} = \begin{cases} \mathbf{t}_v & \text{if } v \in R, \\ 1 & \text{if } v \notin R. \end{cases}$$

For an automorphic form f on $H(\mathbb{A})$ and a set R of places of k, let f_R be the automorphic form on $H^\circ(\mathbb{A})$ defined by

$$f_R(h) = f(h\mathbf{t}_R)$$

for $h \in H^\circ(\mathbb{A})$.

Let f be a cusp form on $H^\circ(\mathbb{A})$ and let $\varphi \in S(V^2(\mathbb{A}))$. Define Whittaker functions W_f and $W_{\theta(f,\varphi)}$ by

$$W_f(h) = \int_{(k\backslash\mathbb{A})^2} f([\mathbf{n}(x), \mathbf{n}(y)]h)\overline{\psi(x+y)}\, dx\, dy,$$
$$W_{\theta(f,\varphi)}(g) = \int_{U(k)\backslash U(\mathbb{A})} \theta(f,\varphi)(ug)\overline{\psi_U(u)}\, du$$

for $g \in G(\mathbb{A})$ and $h \in H^\circ(\mathbb{A})$. Here dx, dy, du are the Tamagawa measures.

Let

$$\Delta N = \{[\mathbf{n}(x), \mathbf{n}(-x)] \in H_1^\circ \mid x \in \mathbb{G}_a\}.$$

In the following lemma, we establish a formula for the Whittaker functions of theta lifts. The result is a variant of [HPS83, (5.18)], which considers Whittaker functions of theta lifts to $\mathrm{Sp}_{2n}(\mathbb{A})$.

Lemma 7.1. *Let $\varphi \in S(V^2(\mathbb{A}))$ and let f be a cusp form on $H(\mathbb{A})$. For $(g,h) \in \mathrm{G}(\mathrm{Sp}_4 \times H_1)(\mathbb{A})$ with $h \in H^\circ(\mathbb{A})$, we have*

$$W_{\theta(f,\varphi)}(g) = 2^{-1-|T|} \sum_{R \subseteq T} \int_{\Delta N(\mathbb{A}) \backslash H_1^\circ(\mathbb{A})} W_{f_R}(h_1 \cdot \mathrm{Ad}(\mathbf{t}_R)h) \omega(g, h_1 \mathbf{t}_R h)\varphi(\mathbf{x}_0, \mathbf{y}_0)\, dh_1,$$

where T is a sufficiently large finite set of places of k, and

$$\mathbf{x}_0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{y}_0 = \mathbf{a}(-1).$$

Proof. Note that

$$\theta(f,\varphi)(g) = 2^{-1} \int_{\boldsymbol{\mu}_2(\mathbb{A})} \int_{H_1^\circ(k)\backslash H_1^\circ(\mathbb{A})} f(h_1 \epsilon h) \Theta(g, h_1 \epsilon h; \varphi)\, dh_1\, d\epsilon.$$

Since

$$\omega(u,1)\varphi(x,y) = \varphi(x, xu_0 + y)\psi(\det(x)u_1 + (x,y)u_2 + \det(y)u_3)$$

for

$$u = \begin{pmatrix} 1 & 0 & u_1 & u_2 \\ 0 & 1 & u_2 & u_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u_0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned} W_{\theta(f,\varphi)}(g) = 2^{-1} \int_{\boldsymbol{\mu}_2(\mathbb{A})} \int_{H_1^\circ(k)\backslash H_1^\circ(\mathbb{A})} \int_{(k\backslash\mathbb{A})^4} f(h_1 \epsilon h) \sum_{x,y \in V(k)} \omega(g, h_1 \epsilon h)\varphi(x, xu_0 + y) \\ \times \psi(\det(x)u_1 + (x,y)u_2 + (\det(y)+1)u_3 + u_0)\, du\, dh_1\, d\epsilon. \end{aligned}$$

Hence, putting

$$\begin{aligned} X_0 &= \left\{(0,y) \in V^2 \mid \det(y) = -1\right\}, \\ X_1 &= \left\{(x,y) \in V^2 \mid \det(x) = 0, x \neq 0, (x,y) = 0, \det(y) = -1\right\}, \end{aligned}$$

and

$$I_i = \int_{\boldsymbol{\mu}_2(\mathbb{A})} \int_{H_1^\circ(k)\backslash H_1^\circ(\mathbb{A})} \int_{k\backslash\mathbb{A}} f(h_1 \epsilon h) \sum_{(x,y) \in X_i(k)} \omega(g, h_1 \epsilon h)\varphi(x, xu_0 + y)\psi(u_0)\, du_0\, dh_1\, d\epsilon,$$

we have

$$W_{\theta(f,\varphi)}(g) = 2^{-1}(I_0 + I_1).$$

In fact, we have

$$I_0 = \int_{\boldsymbol{\mu}_2(\mathbb{A})} \int_{H_1^\circ(k)\backslash H_1^\circ(\mathbb{A})} \int_{k\backslash\mathbb{A}} f(h_1 \epsilon h) \sum_{y \in \mathrm{SL}_2(k)} \omega(g, h_1 \epsilon h)\varphi(0, y\mathbf{a}(-1))\psi(u_0)\, du_0\, dh_1\, d\epsilon = 0$$

since

$$\int_{k\backslash\mathbb{A}} \psi(u_0)\, du_0 = 0.$$

It remains to compute I_1.

We claim that the map $h_1 \mapsto h_1(\mathbf{x}_0, \mathbf{y}_0)$ induces an isomorphism

$$\Delta N \backslash H_1^\circ \longrightarrow X_1.$$

Obviously, H_1° acts on X_1. We first show that this action is transitive. Let $(x,y) \in X_1$ and write

$$y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}.$$

Since the rank of x is 1, there exists $h_1 \in H_1^\circ$ such that $h_1 x = \mathbf{x}_0$. Hence we may assume that $x = \mathbf{x}_0$. Then $(x,y) = y_3 = 0$. Thus we have

$$y = \begin{pmatrix} y_1 & y_2 \\ 0 & -y_1^{-1} \end{pmatrix}.$$

Put

$$h_1 = \left[\begin{pmatrix} 1 & y_1y_2 \\ 0 & -y_1 \end{pmatrix}, \begin{pmatrix} -y_1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Then we have $h_1(\mathbf{x}_0, y) = (\mathbf{x}_0, \mathbf{y}_0)$. We next show that the stabilizer of this action is ΔN in H_1°. Obviously, ΔN fixes $(\mathbf{x}_0, \mathbf{y}_0)$. Assume that $h_1(\mathbf{x}_0, \mathbf{y}_0) = (\mathbf{x}_0, \mathbf{y}_0)$ with $h_1 = [g_1, g_2] \in H_1^\circ$. Then $(g_1\mathbf{x}_0g_2^{-1}, g_1\mathbf{y}_0g_2^{-1}) = (\mathbf{x}_0, \mathbf{y}_0)$. Write

$$g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It follows from $g_1\mathbf{y}_0 = \mathbf{y}_0g_2$ that

$$g_2 = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix},$$

which combined with $g_1\mathbf{x}_0 = \mathbf{x}_0g_2$ implies that $a = d$, $c = 0$. Hence $h_1 \in \Delta N$. This completes the proof of the claim.

If follows from the claim that

$$I_1 = \int_{\boldsymbol{\mu}_2(\mathbb{A})}\int_{\Delta N(\mathbb{A})\backslash H_1^\circ(\mathbb{A})}\int_{\Delta N(k)\backslash\Delta N(\mathbb{A})}\int_{k\backslash\mathbb{A}} f(nh_1\epsilon h)\omega(g, h_1\epsilon h)\varphi(\mathbf{x}_0, \mathbf{x}_0u_0 + \mathbf{y}_0)\psi(u_0)\, du_0\, dn\, dh_1\, d\epsilon.$$

Since

$$\mathbf{x}_0u_0 + \mathbf{y}_0 = \mathbf{n}(-u_0)\mathbf{y}_0,$$

we have

$$\begin{aligned}&\int_{\Delta N(k)\backslash\Delta N(\mathbb{A})}\int_{k\backslash\mathbb{A}} f(nh_1\epsilon h)\omega(g, h_1\epsilon h)\varphi(\mathbf{x}_0, \mathbf{x}_0u_0 + \mathbf{y}_0)\psi(u_0)\, du_0\, dn \\ &= \int_{(k\backslash\mathbb{A})^2} f([\mathbf{n}(x), \mathbf{n}(-x)]h_1\epsilon h)\omega(g, h_1\epsilon h)\varphi(\mathbf{x}_0, \mathbf{n}(-u_0)\mathbf{y}_0)\psi(u_0)\, du_0\, dx.\end{aligned}$$

Note that

$$\omega(g, h_1\epsilon h)\varphi(\mathbf{x}_0, \mathbf{n}(-u_0)\mathbf{y}_0) = \omega(g, [\mathbf{n}(u_0), 1]h_1\epsilon h)\varphi(\mathbf{x}_0, \mathbf{y}_0).$$

Hence I_1 is equal to

$$\begin{aligned}&\int_{\boldsymbol{\mu}_2(\mathbb{A})}\int_{\Delta N(\mathbb{A})\backslash H_1^\circ(\mathbb{A})}\int_{(k\backslash\mathbb{A})^2} f([\mathbf{n}(x-u_0), \mathbf{n}(-x)]h_1\epsilon h)\omega(g, h_1\epsilon h)\varphi(\mathbf{x}_0, \mathbf{y}_0)\psi(u_0)\, du_0\, dx\, dh_1\, d\epsilon \\ &= \int_{\boldsymbol{\mu}_2(\mathbb{A})}\int_{\Delta N(\mathbb{A})\backslash H_1^\circ(\mathbb{A})}\int_{(k\backslash\mathbb{A})^2} f([\mathbf{n}(-u_0), \mathbf{n}(-x)]h_1\epsilon h)\omega(g, h_1\epsilon h)\varphi(\mathbf{x}_0, \mathbf{y}_0)\psi(u_0 + x)\, du_0\, dx\, dh_1\, d\epsilon \\ &= \int_{\boldsymbol{\mu}_2(\mathbb{A})}\int_{\Delta N(\mathbb{A})\backslash H_1^\circ(\mathbb{A})}\int_{(k\backslash\mathbb{A})^2} f([\mathbf{n}(x), \mathbf{n}(y)]h_1\epsilon h)\overline{\psi(x+y)}\omega(g, h_1\epsilon h)\varphi(\mathbf{x}_0, \mathbf{y}_0)\, dx\, dy\, dh_1\, d\epsilon.\end{aligned}$$

This completes the proof. □

Lemma 7.2. *Let $\varphi = \bigotimes_v \varphi_v \in S(V^2(\mathbb{A}))$ be the Schwartz function defined in (6.2)-(6.5). Let $g \in \mathrm{Sp}_4(\mathbb{A})$. We have*

$$W_{\theta(\mathbf{f}^\sharp,\varphi)}(g) = 2^{-1-|(S(\mathrm{DS})\cup S(\mathfrak{n}))\cap\mathfrak{S}|}\cdot\mathfrak{D}^{-5/2}\cdot\xi(2)^{-2}\cdot\left[\prod_v \mathcal{W}_v^+(g_v) + \prod_v \mathcal{W}_v^-(g_v)\right],$$

where

(7.7)
$$\begin{aligned}\mathcal{W}_v^+(g_v) &= \int_{\Delta N(k_v)\backslash H_1^\circ(k_v)} W_v(h_{1,v})\omega_v(g_v, h_{1,v})\varphi_v(\mathbf{x}_0, \mathbf{y}_0)\, d\bar{h}_{1,v}, \\ \mathcal{W}_v^-(g_v) &= \int_{\Delta N(k_v)\backslash H_1^\circ(k_v)} W_v(\mathrm{Ad}(\mathbf{t}_v)h_{1,v})\omega_v(g_v, h_{1,v}\mathbf{t}_v)\varphi_v(\mathbf{x}_0, \mathbf{y}_0)\, d\bar{h}_{1,v}.\end{aligned}$$

Here $d\bar{h}_{1,v}$ is the quotient measure defined by the measures on $\Delta N(k_v)$ and $H_1^\circ(k_v)$ in § 2.2 and § 7.1, respectively.

Proof. Put $S' = (S(\mathrm{DS}) \cup S(\mathfrak{n})) \cap \mathfrak{S}$. Let R be a set of places of k. By (6.1) and the normalization of isomorphisms in (7.6),

$$\mathbf{f}_R^\sharp = \begin{cases} \mathbf{f} & \text{if } S' \cap R = \varnothing, \\ \mathbf{f} \circ \mathrm{Ad}(\mathbf{t}) & \text{if } S' \subseteq R, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$W_{\mathbf{f}_R^\sharp} = \begin{cases} \prod_v W_v & \text{if } S' \cap R = \varnothing, \\ \prod_v W_v \circ \mathrm{Ad}(\mathbf{t}_v) & \text{if } S' \subseteq R, \\ 0 & \text{otherwise.} \end{cases} \tag{7.8}$$

Note that $\omega_v(1, \mathbf{t}_v)\varphi_v = \varphi_v$ for $v \notin S'$. Hence we conclude from (7.8) that for $g \in \mathrm{Sp}_4(\mathbb{A})$, we have

$$\int_{\Delta N(\mathbb{A}) \backslash H_1^\circ(\mathbb{A})} W_{\mathbf{f}_R^\sharp}(h_1)\omega(g, h_1\mathbf{t}_R)\varphi(\mathbf{x}_0, \mathbf{y}_0)\, dh_1 = \mathfrak{D}^{-5/2}\xi(2)^{-2} \cdot \begin{cases} \prod_v \mathcal{W}_v^+(g_v) & \text{if } S' \cap R = \varnothing, \\ \prod_v \mathcal{W}_v^-(g_v) & \text{if } S' \subseteq R, \\ 0 & \text{otherwise.} \end{cases}$$

Here the extra factor $\mathfrak{D}^{-5/2}\xi(2)^{-2}$ is due to the ratio of the Tamagawa measures on $\Delta N(\mathbb{A})$ and $H_1^\circ(\mathbb{A})$ to the corresponding standard measures defined in § 2.2 and § 7.1, respectively. The assertion then follows from Lemma 7.1. This completes the proof. □

7.3. **Calculation of local Whittaker functions.**

7.3.1. *Non-archimedean cases.* Let v be a finite place of k and $\varphi_v \in S(V^2(k_v))$ the Schwartz function defined in (6.2). If $v \nmid \mathfrak{n}$, then

$$\omega_v(k', [k_1, k_2])\varphi_v = \varphi_v \tag{7.9}$$

for $k' \in \mathrm{diag}(1, 1, \varpi_v^{\mathfrak{c}_v}, \varpi_v^{\mathfrak{c}_v})G(\mathfrak{o}_v)\mathrm{diag}(1, 1, \varpi_v^{-\mathfrak{c}_v}, \varpi_v^{-\mathfrak{c}_v})$ and $(k_1, k_2) \in \mathrm{GL}_2(\mathfrak{o}_v) \times \mathrm{GL}_2(\mathfrak{o}_v)$ such that $\nu(k') = \det(k_1k_2^{-1})$. If $v \mid \mathfrak{n}_1$, then

$$\omega_v(k', [k_1, k_2])\varphi_v = \varphi_v \tag{7.10}$$

for $k' \in \mathrm{diag}(1, 1, \varpi_v^{\mathfrak{c}_v}, \varpi_v^{\mathfrak{c}_v})\mathrm{K}(\varpi_v)\mathrm{diag}(1, 1, \varpi_v^{-\mathfrak{c}_v}, \varpi_v^{-\mathfrak{c}_v})$ and $(k_1, k_2) \in K_0(\varpi_v) \times \mathrm{GL}_2(\mathfrak{o}_v)$ such that $\nu(k') = \det(k_1k_2^{-1})$. If $v \mid \mathfrak{n}_2$, then

$$\omega_v(k', [k_1, k_2])\varphi_v = \varphi_v \tag{7.11}$$

for $k' \in \mathrm{diag}(1, 1, \varpi_v^{\mathfrak{c}_v}, \varpi_v^{\mathfrak{c}_v})\mathrm{K}(\varpi_v)\mathrm{diag}(1, 1, \varpi_v^{-\mathfrak{c}_v}, \varpi_v^{-\mathfrak{c}_v})$ and $(k_1, k_2) \in \mathrm{GL}_2(\mathfrak{o}_v) \times K_0(\varpi_v)$ such that $\nu(k') = \det(k_1k_2^{-1})$.

Let $\mathcal{W}_v^\pm$ be the local Whittaker functions defined in (7.7).

Lemma 7.3. *Assume v is a finite place of k and $v \nmid \mathfrak{n}$. We have*

$$\mathcal{W}_v^+(\mathrm{diag}(\varpi_v^{-\mathfrak{c}_v}, 1, \varpi_v^{\mathfrak{c}_v}, 1)) = \mathcal{W}_v^-(\mathrm{diag}(\varpi_v^{-\mathfrak{c}_v}, 1, \varpi_v^{\mathfrak{c}_v}, 1)) = q_v^{\mathfrak{c}_v}.$$

Proof. We drop the subscript v for brevity. Let

$$g = \mathrm{diag}(\varpi^{-\mathfrak{c}}, 1, \varpi^{\mathfrak{c}}, 1) \in \mathrm{Sp}_4(k_v), \quad h = [\mathbf{d}(\varpi^{\mathfrak{c}}), \mathbf{d}(\varpi^{\mathfrak{c}})] \in H_1^\circ(k_v).$$

By (7.9) and our normalization of the Haar measure on $H_1^\circ(k_v)$ in (7.2), we have

$$\mathcal{W}^+(g) = q^{-2\mathfrak{c}}\left(Z^{(1)} + q \cdot Z^{(2)}\right),$$

where

$$Z^{(1)} = \int_{k_v}\int_{k_v^\times}\int_{k_v^\times} W([1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]h)$$
$$\times\, \omega(g,[1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]h)\varphi(\mathbf{x}_0,\mathbf{y}_0)|y_1|^{-2}|y_2|^{-2}\,d^\times y_1\,d^\times y_2\,dx,$$
$$Z^{(2)} = \int_{k_v}\int_{k_v^\times}\int_{k_v^\times} W([\mathbf{a}(\varpi),\mathbf{a}(\varpi)][1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]h)$$
$$\times\, \omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)][1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]h)\varphi(\mathbf{x}_0,\mathbf{y}_0)|y_1|^{-2}|y_2|^{-2}\,d^\times y_1\,d^\times y_2\,dx.$$

For $h_1 \in H_1(k_v)$, we have

$$\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) = q^{2\mathfrak{c}}\cdot\varphi(h_1^{-1}\cdot(\varpi^{-\mathfrak{c}}\mathbf{x}_0,\mathbf{y}_0)).$$

Let $h_1 = [1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]h$ with $x\in k_v$, $y_1,y_2\in k_v^\times$. Then

$$\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) = q^{2\mathfrak{c}}\cdot\varphi\left(\begin{pmatrix}0 & -y_1^{-1}y_2^{-1}\\ 0 & 0\end{pmatrix},\begin{pmatrix}-y_1^{-1}y_2 & -y_1^{-1}y_2^{-1}x\varpi^{\mathfrak{c}}\\ 0 & y_1y_2^{-1}\end{pmatrix}\right),$$
$$\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) = q^{2\mathfrak{c}}\cdot\varphi\left(\begin{pmatrix}0 & -y_1^{-1}y_2^{-1}\varpi^{-1}\\ 0 & 0\end{pmatrix},\begin{pmatrix}-y_1^{-1}y_2 & -y_1^{-1}y_2^{-1}x\varpi^{\mathfrak{c}}\\ 0 & y_1y_2^{-1}\end{pmatrix}\right).$$

Therefore,

$$\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)\neq 0 \text{ if and only if } y_1y_2^{-1}\in\mathfrak{o}^\times, y_1^{-1}\in\mathfrak{o}, \text{ and } x\in y_1y_2\varpi^{-\mathfrak{c}}\mathfrak{o},$$
$$\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)\neq 0 \text{ if and only if } y_1y_2^{-1}\in\mathfrak{o}^\times, y_1^{-1}\in\varpi\mathfrak{o}, \text{ and } x\in y_1y_2\varpi^{-\mathfrak{c}}\mathfrak{o}.$$

On the other hand, the functions $W_1(\mathbf{a}(y))$ and $W_2(\mathbf{a}(y))$ are both supported in $\varpi^{-\mathfrak{c}}\mathfrak{o}$. We conclude that

$$W(h_1)\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)\neq 0 \text{ if and only if } y_1,y_2\in\mathfrak{o}^\times \text{ and } x\in\varpi^{-\mathfrak{c}}\mathfrak{o},$$
$$W([\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) = 0.$$

Hence

$$Z^{(1)} = q^{2\mathfrak{c}}\cdot W([\mathbf{a}(\varpi^{-\mathfrak{c}}),\mathbf{a}(\varpi^{-\mathfrak{c}})])\cdot\mathrm{vol}(\varpi^{-\mathfrak{c}}\mathfrak{o})\mathrm{vol}(\mathfrak{o}^\times)^2 = q^{3\mathfrak{c}},$$
$$Z^{(2)} = 0.$$

A similar calculation shows that $\mathcal{W}^-(g) = \mathcal{W}^+(g)$. This completes the proof. □

Lemma 7.4. *Assume $v \mid \mathfrak{n}$. We have*

$$\mathcal{W}_v^+(\mathrm{diag}(\varpi_v^{-\mathfrak{c}_v},1,\varpi_v^{\mathfrak{c}_v},1)) = \mathcal{W}_v^-(\mathrm{diag}(\varpi_v^{-\mathfrak{c}_v},1,\varpi_v^{\mathfrak{c}_v},1)) = q_v^{\mathfrak{c}_v}(1+q_v)^{-1}.$$

Proof. The calculations for $v\mid\mathfrak{n}_1$ and $v\mid\mathfrak{n}_2$ are similar and we assume $v\mid\mathfrak{n}_1$. We drop the subscript v for brevity. Let $\mathcal{U}$ be the open compact subgroup of $H_1^\circ(k_v)$ defined by

$$\mathcal{U} = H_1^\circ(k_v)\cap(K_0(\varpi)\times\mathrm{GL}_2(\mathfrak{o}))/\mathfrak{o}^\times.$$

Note that

$$\mathcal{C} = \{[k(a),1],[w,1] \mid a\in\mathfrak{o}/\varpi\mathfrak{o}\}$$

is a complete set of coset representatives for $H_1^\circ(\mathfrak{o})/\mathcal{U}$, where

$$k(a) = \begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}.$$

Let

$$g = \mathrm{diag}(\varpi^{-\mathfrak{c}},1,\varpi^{\mathfrak{c}},1)\in\mathrm{Sp}_4(k_v),\quad h = [\mathbf{d}(\varpi^{\mathfrak{c}}),\mathbf{d}(\varpi^{\mathfrak{c}})]\in H_1^\circ(k_v).$$

By (7.10) and our normalization of the Haar measure on $H_1^\circ(k_v)$ in (7.2), we have

$$\mathcal{W}^+(g) = q^{-2\mathfrak{c}}(1+q)^{-1}\left(Z^{(1)} + q\cdot Z^{(2)}\right),$$

where

$$\begin{aligned}
Z^{(1)} &= \sum_{k\in\mathcal{C}} \int_{k_v}\int_{k_v^\times}\int_{k_v^\times} W([1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]hk) \\
&\quad \times \omega(g,[1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]hk)\varphi(\mathbf{x}_0,\mathbf{y}_0)|y_1|^{-2}|y_2|^{-2}\,d^\times y_1\,d^\times y_2\,dx, \\
Z^{(2)} &= \sum_{k\in\mathcal{C}} \int_{k_v}\int_{k_v^\times}\int_{k_v^\times} W([\mathbf{a}(\varpi),\mathbf{a}(\varpi)][1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]hk) \\
&\quad \times \omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)][1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]hk)\varphi(\mathbf{x}_0,\mathbf{y}_0)|y_1|^{-2}|y_2|^{-2}\,d^\times y_1\,d^\times y_2\,dx.
\end{aligned}$$

For $h_1 \in H_1(k_v)$, we have

$$\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) = q^{2\mathfrak{c}}\cdot\varphi(h_1^{-1}\cdot(\varpi^{-\mathfrak{c}}\mathbf{x}_0,\mathbf{y}_0)).$$

Let $h_1 = [1,\mathbf{n}(x)][\mathbf{m}(y_1),\mathbf{m}(y_2)]hk$ with $x\in k_v$, $y_1,y_2\in k_v^\times$, and $k\in\mathcal{C}$. If $k=[k(a),1]$, then

$$\begin{aligned}
\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) &= q^{2\mathfrak{c}}\cdot\varphi\left(\begin{pmatrix}0 & -y_1^{-1}y_2^{-1}\\ 0 & ay_1^{-1}y_2^{-1}\end{pmatrix},\begin{pmatrix}-y_1^{-1}y_2 & -y_1^{-1}y_2^{-1}x\varpi^{\mathfrak{c}}\\ ay_1^{-1}y_2 & y_1y_2^{-1}+ay_1^{-1}y_2^{-1}x\varpi^{\mathfrak{c}}\end{pmatrix}\right),\\
\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) &= q^{2\mathfrak{c}}\cdot\varphi\left(\begin{pmatrix}0 & -y_1^{-1}y_2^{-1}\varpi^{-1}\\ 0 & ay_1^{-1}y_2^{-1}\varpi^{-1}\end{pmatrix},\begin{pmatrix}-y_1^{-1}y_2 & -y_1^{-1}y_2^{-1}x\varpi^{\mathfrak{c}}\\ ay_1^{-1}y_2 & y_1y_2^{-1}+ay_1^{-1}y_2^{-1}x\varpi^{\mathfrak{c}}\end{pmatrix}\right).
\end{aligned}$$

If $k=[w,1]$, then

$$\begin{aligned}
\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) &= q^{2\mathfrak{c}}\cdot\varphi\left(\begin{pmatrix}0 & 0\\ 0 & -y_1^{-1}y_2^{-1}\end{pmatrix},\begin{pmatrix}0 & -y_1y_2^{-1}\\ -y_1^{-1}y_2 & -y_1^{-1}y_2^{-1}x\varpi^{\mathfrak{c}}\end{pmatrix}\right),\\
\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) &= q^{2\mathfrak{c}}\cdot\varphi\left(\begin{pmatrix}0 & 0\\ 0 & -y_1^{-1}y_2^{-1}\varpi^{-1}\end{pmatrix},\begin{pmatrix}0 & -y_1y_2^{-1}\\ -y_1^{-1}y_2 & -y_1^{-1}y_2^{-1}x\varpi^{\mathfrak{c}}\end{pmatrix}\right).
\end{aligned}$$

Therefore, if $k=1$, then

$$\begin{aligned}
\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)\neq 0 &\text{ if and only if } y_1y_2^{-1}\in\mathfrak{o}^\times, y_1^{-1}\in\mathfrak{o},\ \text{ and } x\in y_1y_2\varpi^{-\mathfrak{c}}\mathfrak{o},\\
\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)\neq 0 &\text{ if and only if } y_1y_2^{-1}\in\mathfrak{o}^\times, y_1^{-1}\in\varpi\mathfrak{o},\ \text{ and } x\in y_1y_2\varpi^{-\mathfrak{c}}\mathfrak{o},
\end{aligned}$$

and if $k\neq 1$, then

$$\begin{aligned}
\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)\neq 0 &\text{ if and only if } y_1y_2^{-1}\in\mathfrak{o}^\times, y_1^{-1}\in\varpi\mathfrak{o},\ \text{ and } x\in y_1y_2\varpi^{-\mathfrak{c}}\mathfrak{o},\\
\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)\neq 0 &\text{ if and only if } y_1y_2^{-1}\in\mathfrak{o}^\times, y_1^{-1}\in\varpi\mathfrak{o},\ \text{ and } x\in y_1y_2\varpi^{-\mathfrak{c}}\mathfrak{o}.
\end{aligned}$$

On the other hand, the functions $W_1(\mathbf{a}(y))$ and $W_2(\mathbf{a}(y))$ are both supported in $\varpi^{-\mathfrak{c}}\mathfrak{o}$, whereas the functions $W_1(\mathbf{a}(y)k(a))$ with $a\in\mathfrak{o}^\times$ and $W_1(\mathbf{a}(y)w)$ are both supported in $\varpi^{-\mathfrak{c}-1}\mathfrak{o}$. We conclude that if $k=1$, then

$$\begin{aligned}
&W(h_1)\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)\neq 0 \text{ if and only if } y_1,y_2\in\mathfrak{o}^\times \text{ and } x\in\varpi^{-\mathfrak{c}}\mathfrak{o},\\
&W([\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0)=0,
\end{aligned}$$

and if $k\neq 1$, then

$$\begin{aligned}
W(h_1)\omega(g,h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) &= 0,\\
W([\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\omega(g,[\mathbf{a}(\varpi),\mathbf{a}(\varpi)]h_1)\varphi(\mathbf{x}_0,\mathbf{y}_0) &= 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&Z^{(1)} = q^{2\mathfrak{c}}\cdot W([\mathbf{a}(\varpi^{-\mathfrak{c}}),\mathbf{a}(\varpi^{-\mathfrak{c}})])\cdot\mathrm{vol}(\varpi^{-\mathfrak{c}}\mathfrak{o})\mathrm{vol}(\mathfrak{o}^\times)^2 = q^{3\mathfrak{c}},\\
&Z^{(2)} = 0.
\end{aligned}$$

A similar calculation shows that $\mathcal{W}^-(g)=\mathcal{W}^+(g)$. This completes the proof. □

7.3.2. *Archimedean cases.* Let v be a real place of k. We identify k_v with $\mathbb{R}$. Let $\varphi_v \in S(V^2(\mathbb{R}))$ be the Schwartz function defined in (6.3)-(6.5). If $v \in S(\mathrm{DS})$, then

$$\omega_v(1, [k_{\theta_1}, k_{\theta_2}])\varphi_v = e^{-\sqrt{-1}(\kappa_{1,v}\theta_1 + \kappa_{2,v}\theta_2)}\varphi_v \tag{7.12}$$

for $k_{\theta_1}, k_{\theta_2} \in \mathrm{SO}(2)$, and

$$Z \cdot \varphi_v = -\lambda_1 \cdot \varphi_v, \quad Z' \cdot \varphi_v = -\lambda_2 \cdot \varphi_v, \quad N_- \cdot \varphi_v = 0. \tag{7.13}$$

Here Z, Z', N_- are elements in $\mathfrak{sp}_4(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ defined by

$$Z = -\sqrt{-1}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z' = -\sqrt{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad N_- = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & \sqrt{-1} \\ -1 & 0 & \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} & 0 & 1 \\ -\sqrt{-1} & 0 & -1 & 0 \end{pmatrix}.$$

If $v \in S(\mathrm{PS})$, then

$$\omega_v(k, [k_1, k_2])\varphi_v = \varphi_v \tag{7.14}$$

for $k \in G(\mathbb{R}) \cap \mathrm{O}(4)$ and $k_1, k_2 \in \mathrm{O}(2)$ with $\nu(k) = \det(k_1 k_2^{-1})$.

For $n \in \mathbb{Z}_{\geq 0}$, let

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

denote the Hermite polynomial.

Lemma 7.5. *Let $r_1, r_2, r_3 \in \mathbb{R}$ with $r_1 > 0$ and $r_2^2 + r_3 > 0$. For $n \in \mathbb{Z}_{\geq 0}$, put*

$$J_n(r_1, r_2, r_3) = \int_0^\infty y^{n-2} H_n(\sqrt{\pi}(r_1 y + r_2 y^{-1})) e^{-\pi(r_1 y + r_2 y^{-1})^2 - \pi r_3 y^{-2}}\, dy.$$

Then

$$J_n(r_1, r_2, r_3) = 2^{n-1}\pi^{n/2}\left((r_2^2 + r_3)^{1/2} + r_2\right)^n (r_2^2 + r_3)^{-1/2} e^{-2\pi r_1\left((r_2^2+r_3)^{1/2} + r_2\right)}.$$

Proof. The assertion is a variant of [Ich05, Lemma 7.5]. In a small enough neighborhood of $x = 0$, we have

$$\begin{aligned} \sum_{n=0}^\infty \frac{1}{n!}(-\sqrt{\pi}x)^n J_n(r_1, r_2, r_3) &= \int_0^\infty y^{-2} e^{-\pi(xy + r_1 y + r_2 y^{-1})^2 - \pi r_3 y^{-2}}\, dy \\ &= 2^{-1}(r_2^2 + r_3)^{-1/2} e^{-2\pi r_2(x + r_1)} e^{-2\pi(r_2^2 + r_3)^{1/2}|x + r_1|} \\ &= 2^{-1}(r_2^2 + r_3)^{-1/2} e^{-2\pi\left((r_2^2+r_3)^{1/2} + r_2\right)(x + r_1)}. \end{aligned}$$

This completes the proof. □

Lemma 7.6. *Let $n \in \mathbb{Z}_{\geq 0}$. Put*

$$\begin{aligned} h_n(a_1, a_2) = a_1 a_2^{n+1} \int_0^\infty & y^{-n-2}\left((a_1^2 y^{-2} + a_2^2 y^2)^{1/2} - a_2 y\right)^n (a_1^2 y^{-2} + a_2^2 y^2)^{-1/2} \\ & \times e^{-2\pi\left(y^2 - a_2^2 + (a_2 y^{-1} + a_2^{-1} y)(a_1^2 y^{-2} + a_2^2 y^2)^{1/2}\right)}\, dy \end{aligned}$$

for $a_1, a_2 > 0$. Let $s_1, s_2 \in \mathbb{C}$ satisfy $\mathrm{Re}(s_1 + s_2 + 1) > 0$ and $\mathrm{Re}(s_1) > 0 > \mathrm{Re}(s_2)$. Then

$$\begin{aligned} &\int_0^\infty \int_0^\infty x_1^{s_1 - 1} x_2^{s_2 - 1} h_n(x_1, x_2)\, dx_1\, dx_2 \\ &= 2^{-n-4}\pi^{-n-1}(4\pi^3)^{-s_1/2}(4\pi)^{-s_2/2}\Gamma\left(\frac{s_1 + s_2 + 2n + 1}{2}\right)\Gamma\left(\frac{s_1 + s_2 + 1}{2}\right)\Gamma\left(\frac{s_1}{2}\right)\Gamma\left(\frac{-s_2}{2}\right). \end{aligned}$$

Proof. We make a change of variable from x_2 to $x_1 y^{-2} x_2$. Then

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x_1^{s_1-1} x_2^{s_2-1} h_n(x_1,x_2)\,dx_1\,dx_2 \\
&= \int_0^\infty \int_0^\infty \int_0^\infty x_1^{s_1+s_2+2n} x_2^{s_2+n} y^{-2s_2-4n-3} \left((1+x_2^2)^{1/2}-x_2\right)^n (1+x_2^2)^{-1/2} \\
&\qquad \times e^{-2\pi x_1^2 x_2 y^{-4}\left((1+x_2^2)^{1/2}-x_2\right)} e^{-2\pi y^2 x_2^{-1}\left((1+x_2^2)^{1/2}+x_2\right)}\,dx_1\,dy\,dx_2 \\
&= 2^{-1}(2\pi)^{-(s_1+s_2+2n+1)/2}\Gamma\left(\frac{s_1+s_2+2n+1}{2}\right) \\
&\times \int_0^\infty \int_0^\infty x_2^{(-s_1+s_2-1)/2} y^{2s_1-1} \left((1+x_2^2)^{1/2}-x_2\right)^{-(s_1+s_2+1)/2} (1+x_2^2)^{-1/2} \\
&\qquad \times e^{-2\pi y^2 x_2^{-1}\left((1+x_2^2)^{1/2}+x_2\right)}\,dy\,dx_2 \\
&= 2^{-2}(2\pi)^{(-3s_1-s_2-2n-1)/2}\Gamma\left(\frac{s_1+s_2+2n+1}{2}\right)\Gamma(s_1) \\
&\times \int_0^\infty x_2^{(s_1+s_2-1)/2} \left((1+x_2^2)^{1/2}+x_2\right)^{(-s_1+s_2+1)/2} (1+x_2^2)^{-1/2}\,dx_2 \\
&= 2^{-n-4}\pi^{-n-1}(4\pi^3)^{-s_1/2}(4\pi)^{-s_2/2}\Gamma\left(\frac{s_1+s_2+2n+1}{2}\right)\Gamma\left(\frac{s_1+s_2+1}{2}\right)\Gamma\left(\frac{s_1}{2}\right)\Gamma\left(\frac{-s_2}{2}\right).
\end{aligned}$$

Here the last equality follows from [BE54, p. 311, (28)] and the duplication formula

$$\Gamma\left(\frac{s_1}{2}\right)\Gamma\left(\frac{s_1+1}{2}\right) = 2^{1-s_1}\sqrt{\pi}\,\Gamma(s_1).$$

This completes the proof. □

Lemma 7.7. *Let $v \in S(\mathrm{DS})$. For $a_1, a_2 > 0$, we have*

$$\begin{aligned}
&\mathcal{W}_v^+(\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})) \\
&= \mathcal{W}_v^-(\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})) \\
&= 2^{-\lambda_{1,v}-4}\pi^{(-3\lambda_{1,v}+\lambda_{2,v}-5)/2} \\
&\times e^{-2\pi a_2^2} \int_{c_1-\sqrt{-1}\infty}^{c_1+\sqrt{-1}\infty} \frac{ds_1}{2\pi\sqrt{-1}} \int_{c_2-\sqrt{-1}\infty}^{c_2+\sqrt{-1}\infty} \frac{ds_2}{2\pi\sqrt{-1}} (4\pi^3 a_1^2)^{(-s_1+\lambda_{1,v}+1)/2} (4\pi a_2^2)^{(-s_2+\lambda_{2,v})/2} \\
&\qquad \times \Gamma\left(\frac{s_1+s_2-2\lambda_{2,v}+1}{2}\right)\Gamma\left(\frac{s_1+s_2+1}{2}\right)\Gamma\left(\frac{s_1}{2}\right)\Gamma\left(\frac{-s_2}{2}\right).
\end{aligned}$$

Here $c_1, c_2 \in \mathbb{R}$ satisfy

$$c_1+c_2+1>0, \quad c_1>0>c_2.$$

Proof. We identify $k_v = \mathbb{R}$ and drop the subscript v for brevity. Without loss of generality, we assume $\kappa_1 \geq \kappa_2$. Let $g = \mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})$. Note that $W|_{H_1^\circ(\mathbb{R})}$ is supported in $H_1^\circ(\mathbb{R})^0$ and

$$W([\mathbf{a}(y_1),\mathbf{a}(y_2)]) = y_1^{\kappa_1/2} y_2^{\kappa_2/2} e^{-2\pi y_1 - 2\pi y_2}$$

for $y_1, y_2 > 0$. By (7.3) and (7.12), we have

$$\begin{aligned}
\mathcal{W}^+(g) = 4a_1^{\lambda_1+2}a_2^{-\lambda_2+2} &\int_0^\infty \int_0^\infty y_1^{-\lambda_2-2} y_2^{\lambda_2-2} e^{-\pi(a_1^2 y_1^{-2} y_2^{-2} + a_2^2 y_1^{-2} y_2^2 + a_2^2 y_1^2 y_2^{-2} + 2y_1^2 + 2y_2^2)} \\
&\times \int_{\mathbb{R}} (y_1 y_2^{-1} - y_1^{-1} y_2 - \sqrt{-1}x)^{-\lambda_2} e^{-\pi a_2^2 x^2} e^{2\pi\sqrt{-1}x y_1 y_2}\,dx\,dy_1\,dy_2.
\end{aligned}$$

By [Ich05, Lemma 7.4], we have

$$\begin{aligned}
&\int_{\mathbb{R}} (y_1 y_2^{-1} - y_1^{-1} y_2 - \sqrt{-1}x)^{-\lambda_2} e^{-\pi a_2^2 x^2} e^{2\pi\sqrt{-1}x y_1 y_2}\,dx \\
&= (2\sqrt{\pi})^{\lambda_2} a_2^{\lambda_2-1} H_{-\lambda_2}(\sqrt{\pi}(a_2 y_1 y_2^{-1} - a_2 y_1^{-1} y_2 + a_2^{-1} y_1 y_2)) e^{-\pi a_2^{-2} y_1^2 y_2^2}.
\end{aligned}$$

Therefore, by Lemma 7.5,

$$\begin{aligned}\mathcal{W}^+(g) &= 4(2\sqrt{\pi})^{\lambda_2} a_1^{\lambda_1+2} a_2 \int_0^\infty \int_0^\infty y_1^{-\lambda_2-2} y_2^{\lambda_2-2} e^{-\pi(a_1^2 y_1^{-2} y_2^{-2} + a_2^2 y_1^{-2} y_2^2 + a_2^2 y_1^2 y_2^{-2} + a_2^{-2} y_1^2 y_2^2 + 2y_1^2 + 2y_2^2)} \\ &\qquad \times H_{-\lambda_2}(\sqrt{\pi}(a_2 y_1 y_2^{-1} - a_2 y_1^{-1} y_2 + a_2^{-1} y_1 y_2))\, dy_1\, dy_2 \\ &= 4(2\sqrt{\pi})^{\lambda_2} a_1^{\lambda_1+2} a_2 \int_0^\infty y^{\lambda_2-2} e^{-4\pi y^2 - 2\pi a_2^2} J_{-\lambda_2}(a_2 y^{-1} + a_2^{-1} y,\, -a_2 y,\, a_1^2 y^{-2})\, dy \\ &= 2 a_1^{\lambda_1+1} a_2^{\lambda_2} e^{-2\pi a_2^2} h_{-\lambda_2}(a_1, a_2).\end{aligned}$$

A similar calculation shows that $\mathcal{W}^-(g) = \mathcal{W}^+(g)$. The assertion then follows from Lemma 7.6 and the Mellin inversion formula. This completes the proof. □

Lemma 7.8. *Let $v \in S(\mathrm{PS})$. For $a_1, a_2 > 0$, we have*

$$\begin{aligned}&\mathcal{W}_v^+(\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})) \\ &= \mathcal{W}_v^-(\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})) \\ &= 2^{-4} a_1^2 a_2 \\ &\times \int_{c_1-\sqrt{-1}\infty}^{c_1+\sqrt{-1}\infty} \frac{ds_1}{2\pi\sqrt{-1}} \int_{c_2-\sqrt{-1}\infty}^{c_2+\sqrt{-1}\infty} \frac{ds_2}{2\pi\sqrt{-1}} \left(\pi a_1 a_2^{-1}\right)^{-s_1} \left(\pi a_2^2\right)^{-s_2} \\ &\times \Gamma\left(\frac{s_1+\lambda_{1,v}}{2}\right) \Gamma\left(\frac{s_1-\lambda_{1,v}}{2}\right) \Gamma\left(\frac{s_1+\lambda_{2,v}}{2}\right) \Gamma\left(\frac{s_1-\lambda_{2,v}}{2}\right) \\ &\times \Gamma\left(\frac{s_2}{2} + \frac{\lambda_{1,v}+\lambda_{2,v}}{4}\right) \Gamma\left(\frac{s_2}{2} - \frac{\lambda_{1,v}+\lambda_{2,v}}{4}\right) \Gamma\left(\frac{s_2}{2} + \frac{\lambda_{1,v}-\lambda_{2,v}}{4}\right) \Gamma\left(\frac{s_2}{2} - \frac{\lambda_{1,v}-\lambda_{2,v}}{4}\right) \\ &\times \Gamma\left(\frac{s_1+s_2}{2} + \frac{\lambda_{1,v}+\lambda_{2,v}}{4}\right)^{-1} \Gamma\left(\frac{s_1+s_2}{2} - \frac{\lambda_{1,v}+\lambda_{2,v}}{4}\right)^{-1} \\ &\times {}_3F_2\left(\frac{s_1}{2}, \frac{s_2}{2} + \frac{\lambda_{1,v}-\lambda_{2,v}}{4}, \frac{s_2}{2} - \frac{\lambda_{1,v}-\lambda_{2,v}}{4}; \frac{s_1+s_2}{2} + \frac{\lambda_{1,v}+\lambda_{2,v}}{4}, \frac{s_1+s_2}{2} - \frac{\lambda_{1,v}+\lambda_{2,v}}{4}; 1\right).\end{aligned}$$

Here $c_1, c_2 \in \mathbb{R}$ satisfy

$$c_1 > \max\left\{|\mathrm{Re}(\lambda_{1,v})|,\, |\mathrm{Re}(\lambda_{2,v})|\right\}, \quad c_2 > \max\left\{\left|\mathrm{Re}\left(\frac{\lambda_{1,v}+\lambda_{2,v}}{2}\right)\right|, \left|\mathrm{Re}\left(\frac{\lambda_{1,v}-\lambda_{2,v}}{2}\right)\right|\right\}.$$

Proof. We identify $k_v = \mathbb{R}$ and drop the subscript v for brevity. Let $g = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})$. Note that

$$W([\mathbf{a}(y_1), \mathbf{a}(y_2)]) = \mathrm{sgn}^\varepsilon(y_1 y_2) |y_1 y_2|^{1/2} K_{\mu_1}(2\pi|y_1|) K_{\mu_2}(2\pi|y_2|)$$

for $y_1, y_2 \in \mathbb{R}^\times$. By (7.3) and (7.14), we have

$$\begin{aligned}\mathcal{W}^+(g) &= 8 a_1^2 a_2^2 \int_0^\infty \int_0^\infty y_1^{-2} y_2^{-2} K_{\mu_1}(2\pi y_1^2) K_{\mu_2}(2\pi y_2^2) e^{-\pi(a_1^2 y_1^{-2} y_2^{-2} + a_2^2 y_1^{-2} y_2^2 + a_2^2 y_1^2 y_2^{-2})} \\ &\qquad \times \int_{\mathbb{R}} e^{-\pi a_2^2 y_1^{-2} y_2^{-2} x^2} e^{2\pi\sqrt{-1}x}\, dx\, dy_1\, dy_2 \\ &= 8 a_1^2 a_2 \int_0^\infty \int_0^\infty y_1^{-1} y_2^{-1} K_{\mu_1}(2\pi y_1^2) K_{\mu_2}(2\pi y_2^2) e^{-\pi(a_1^2 y_1^{-2} y_2^{-2} + a_2^2 y_1^{-2} y_2^2 + a_2^2 y_1^2 y_2^{-2} + a_2^{-2} y_1^2 y_2^2)}\, dy_1\, dy_2.\end{aligned}$$

Therefore, in the notation of [Ish05, Theorem 3.2], we have

$$\mathcal{W}^+(g) = W_{(\lambda_1,\lambda_2)}^{(0,0)}((a_1 a_2^{-1}, a_2^2)).$$

A similar calculation shows that $\mathcal{W}^-(g) = \mathcal{W}^+(g)$. The assertion then follows from [Ish05, Proposition 3.5]. □

7.4. **Proof of Proposition 6.1.** Let $g_0, g_1 \in G(\mathbb{A})$ be the elements defined by

$$g_{0,v} = \begin{cases} \mathrm{diag}(1,1,\varpi_v^{\mathfrak{c}_v},\varpi_v^{\mathfrak{c}_v}) & \text{if } v \nmid \infty, \\ 1 & \text{if } v \mid \infty, \end{cases}$$

$$g_{1,v} = \begin{cases} \mathrm{diag}(\varpi_v^{-\mathfrak{c}_v},1,\varpi_v^{\mathfrak{c}_v},1) & \text{if } v \nmid \infty, \\ 1 & \text{if } v \mid \infty. \end{cases}$$

Let $\varphi = \bigotimes_v \varphi_v \in S(V^2(\mathbb{A}))$ be the Schwartz function defined in (6.2)-(6.5). By (7.9)-(7.14), there exists a constant C such that

$$\theta(\mathbf{f}^\sharp, \varphi)(g) = C \cdot f(gg_0)$$

for $g \in G(\mathbb{A})$. To determine C, by the normalization of f in (2.5), it suffices to compute $W_{\theta(\mathbf{f}^\sharp,\varphi)}(g_1)$. The assertion then follows from Lemmas 7.2-7.4, 7.7 and 7.8. This completes the proof of Proposition 6.1.

8. Explicit Rallis inner product formula

We keep the notation of § 6 and § 7. The aim of this section is to prove Proposition 6.2. We prove it by specializing the Rallis inner product formula in [GQT14] to the theta lift $\theta(\mathbf{f}^\sharp, \varphi)$. It boils down to the calculation of certain doubling local zeta integrals involving matrix coefficients of $\sigma_v^\sharp$ and the Weil representation ω_v for each place v.

8.1. Matrix coefficients.

8.1.1. *Hermitian pairings on σ_v and $\sigma_v^\sharp$.* Let v be a place of k. Let $\langle\ ,\ \rangle_{i,v} : \mathcal{V}_{i,v} \otimes \overline{\mathcal{V}}_{i,v} \to \mathbb{C}$ be the $\mathrm{GL}_2(k_v)$-invariant Hermitian pairing defined by

$$\langle W_1, W_2 \rangle_{i,v} = \int_{k_v^\times} W_1(\mathbf{a}(t)) \overline{W_2(\mathbf{a}(t))}\, d^\times t$$

for $W_1, W_2 \in \mathcal{V}_{i,v}$. Let $\mathcal{B}_v : \mathcal{V}_v \otimes \overline{\mathcal{V}}_v \to \mathbb{C}$ be the $H^\circ(k_v)$-invariant Hermitian pairing defined by

$$\mathcal{B}_v(W_1 \otimes W_2, W_3 \otimes W_4) = \frac{\zeta_v(2)^2}{\zeta_v(1)^2} \cdot \frac{\langle W_1, W_3 \rangle_{1,v} \langle W_2, W_4 \rangle_{2,v}}{L(1, \sigma_{1,v}, \mathrm{Ad}) L(1, \sigma_{2,v}, \mathrm{Ad})}$$

for $W_1, W_3 \in \mathcal{V}_{1,v}$ and $W_2, W_4 \in \mathcal{V}_{2,v}$. Let $\mathcal{B}_v^\sharp : \mathcal{V}_v^\sharp \otimes \overline{\mathcal{V}}_v^\sharp \to \mathbb{C}$ be the $H(k_v)$-invariant Hermitian pairing defined as follows:

- If $v \notin \mathfrak{S}$, then $\mathcal{B}_v^\sharp = \mathcal{B}_v$.
- If $v \in \mathfrak{S}$, then

$$\mathcal{B}_v^\sharp((W_1, W_2), (W_3, W_4)) = \frac{1}{2} \left[\mathcal{B}_v(W_1, W_3) + \mathcal{B}_v(W_2, W_4) \right]$$

  for (W_1, W_2), $(W_3, W_4) \in \mathcal{V}_v^\sharp$.

8.1.2. *Matrix coefficients of σ_v.* Let v be a place of k. Let W_v be the Whittaker function on $H^\circ(k_v)$ defined in (7.5).

Lemma 8.1. *Let $v \nmid \infty\mathfrak{n}$. We have*

$$\mathcal{B}_v(W_v, W_v) = 1.$$

Proof. The calculation was carried out in [IP18, Lemma 6.9]. □

Lemma 8.2. *Let $v \mid \mathfrak{n}_1$. Write*

$$\sigma_{1,v} = \mathrm{St} \otimes \eta, \quad \sigma_{2,v} = \mathrm{Ind}_{B(k_v)}^{\mathrm{GL}_2(k_v)}(\chi \boxtimes \chi^{-1}),$$

where St *is the Steinberg representation of* $\mathrm{GL}_2(k_v)$*, and χ and η are unramified characters of $k_v^\times$ such that $\chi = |\ |_v^s$ with $|\mathrm{Re}(s)| < 1/2$ and $\eta^2 = 1$. For*

$$h_{n,m} = [\mathbf{a}(\varpi_v^n)\mathbf{d}(\varpi_v^m), \mathbf{a}(\varpi_v^{n+m})], \quad h'_{n,m} = [w\mathbf{a}(\varpi_v^n)\mathbf{d}(\varpi_v^m), \mathbf{a}(\varpi_v^{n+m})]$$

with $m, n \in \mathbb{Z}$ such that $m+n \geq 0$, we have

$$\mathcal{B}_v(\sigma_v(h_{n,m})W_v, W_v) = \frac{\zeta_v(2)}{\zeta_v(1)} \cdot \varepsilon^{n-m} \cdot \frac{q_v^{-|n-m|-(n+m)/2}}{1+q_v^{-1}} \cdot \left(\alpha^{n+m} \cdot \frac{1-\alpha^{-2}q_v^{-1}}{1-\alpha^{-2}} + \alpha^{-(n+m)} \cdot \frac{1-\alpha^2 q_v^{-1}}{1-\alpha^2}\right),$$

$$\mathcal{B}_v(\sigma_v(h'_{n,m})W_v, W_v) = -\frac{\zeta_v(2)}{\zeta_v(1)} \cdot \varepsilon^{n-m} \cdot \frac{q_v^{-|n-m-1|-(n+m)/2}}{1+q_v^{-1}} \cdot \left(\alpha^{n+m} \cdot \frac{1-\alpha^{-2}q_v^{-1}}{1-\alpha^{-2}} + \alpha^{-(n+m)} \cdot \frac{1-\alpha^2 q_v^{-1}}{1-\alpha^2}\right),$$

where

$$\alpha = \chi(\varpi_v), \quad \varepsilon = \eta(\varpi_v). \tag{8.1}$$

Proof. By [IP18, Lemmas 6.9 and 6.11],

$$\mathcal{B}_v(W_v, W_v) = \frac{\zeta_v(2)}{\zeta_v(1)}.$$

The assertion then follows from the formulae for matrix coefficients in [Mac71] and [GJ72, §7]. □

Lemma 8.3. *Let $v \in S(\mathrm{DS})$. For $h = [k_{\theta_1}, k_{\theta_2}][\mathbf{m}(e^{t_1}), \mathbf{m}(e^{t_2})][k_{\theta_3}, k_{\theta_4}] \in H_1^\circ(\mathbb{R})^0$ with $k_{\theta_1}, k_{\theta_2}, k_{\theta_3}, k_{\theta_4} \in \mathrm{SO}(2)$ and $t_1, t_2 \in \mathbb{R}$, we have*

$$\mathcal{B}_v(\sigma_v(h)W_v, W_v) = 2^{-2\lambda_{1,v}-2} e^{\sqrt{-1}(\kappa_{1,v}\theta_1+\kappa_{1,v}\theta_3+\kappa_{2,v}\theta_2+\kappa_{2,v}\theta_4)} \cosh(t_1)^{-\kappa_{1,v}} \cosh(t_2)^{-\kappa_{2,v}}.$$

For $h \in H^\circ(\mathbb{R}) \setminus H^\circ(\mathbb{R})^0$, we have $\mathcal{B}_v(\sigma_v(h)W_v, W_v) = 0$.

Proof. The assertion follows from the formulae for the Whittaker functions that

$$W_{i,v}(\mathbf{a}(y)k_\theta) = \begin{cases} e^{\sqrt{-1}\kappa_{i,v}\theta} y^{\kappa_{i,v}/2} e^{-2\pi y} & \text{if } y > 0, \\ 0 & \text{if } y < 0, \end{cases}$$

for $i = 1, 2$, $y \in \mathbb{R}^\times$ and $k_\theta \in \mathrm{SO}(2)$. □

8.1.3. *Matrix coefficients of the Weil representations.* Let v be a place of k and $\varphi_v \in S(V^2(k_v))$ the Schwartz function defined in (6.2)-(6.5). Let Φ_v be the matrix coefficient of the Weil representation ω_v defined by

$$\Phi_v(h_{1,v}) = \int_{V^2(k_v)} \omega_v(1, h_{1,v})\varphi_v(x_v)\overline{\varphi_v(x_v)}\, dx_v$$

for $h_{1,v} \in H_1(k_v)$.

Lemma 8.4. *Let $v \in S(\mathrm{PS})$. We have*

$$\mathcal{B}_v(W_v, W_v) = 2^{-4}.$$

Proof. By [GR15, 6.576.4], we have

$$\langle W_{i,v}, W_{i,v}\rangle_{i,v} = \int_{\mathbb{R}} K_{\mu_{i,v}}(2\pi|t|)^2\, dt = 2^{-2}\Gamma\left(\frac{1}{2}+\mu_{i,v}\right)\Gamma\left(\frac{1}{2}-\mu_{i,v}\right)$$

for $i = 1, 2$. Therefore

$$\mathcal{B}_v(W_v, W_v) = \frac{\zeta_v(2)^2}{\zeta_v(1)^2} \cdot \frac{\langle W_{1,v}, W_{1,v}\rangle_{1,v}\langle W_{2,v}, W_{2,v}\rangle_{2,v}}{L(1, \sigma_{1,v}, \mathrm{Ad})L(1, \sigma_{2,v}, \mathrm{Ad})} = 2^{-4}.$$

□

Lemma 8.5. *Let v be a finite place of k. Let $n, m \in \mathbb{Z}$. If $v \nmid \mathfrak{n}$, then*

$$\Phi_v([\mathbf{a}(\varpi_v^n)\mathbf{d}(\varpi_v^m), \mathbf{a}(\varpi_v^{n+m})]) = q_v^{-2|n|-2|m|}.$$

If $v \mid \mathfrak{n}_1$, then

$$\Phi_v([\mathbf{a}(\varpi_v^n)\mathbf{d}(\varpi_v^m), \mathbf{a}(\varpi_v^{n+m})]) = q_v^{-2|n|-2|m|-2},$$

$$\Phi_v([w\mathbf{a}(\varpi_v^n)\mathbf{d}(\varpi_v^m), \mathbf{a}(\varpi_v^{n+m})]) = q_v^{-|n|-|n-1|-|m|-|m+1|-2}.$$

If $v \mid \mathfrak{n}_2$, then

$$\Phi_v([\mathbf{a}(\varpi_v^{n+m}), \mathbf{a}(\varpi_v^n)\mathbf{d}(\varpi_v^m)]) = q_v^{-2|n|-2|m|-2},$$

$$\Phi_v([\mathbf{a}(\varpi_v^{n+m}), w\mathbf{a}(\varpi_v^n)\mathbf{d}(\varpi_v^m)]) = q_v^{-|n|-|n-1|-|m|-|m+1|-2}.$$

Proof. The verification is straightforward and we leave it to the readers. □

Lemma 8.6. *Let $v \in S(\mathrm{DS})$. For $a, b > 0$, we have*

$$\Phi_v([\mathbf{m}(a), \mathbf{m}(b)]) = \pi^{-\lambda_{1,v}+\lambda_{2,v}}\Gamma(\lambda_{1,v}+1)\Gamma(-\lambda_{2,v}+1)(ab+a^{-1}b^{-1})^{-2}(ab^{-1}+a^{-1}b)^{-2} \times \left[(ab+a^{-1}b^{-1})^{-1}+(ab^{-1}+a^{-1}b)^{-1}\right]^{\lambda_{1,v}-\lambda_{2,v}}.$$

Proof. We identify $k_v = \mathbb{R}$ and drop the subscript v for brevity. Without loss of generality, we assume $\kappa_1 \geq \kappa_2$. For $n \in \mathbb{Z}_{\geq 0}$ and $z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, let

$$f_n(z) = \int_{\mathbb{R}^4} (a^{-1}x_1 - a^{-1}\sqrt{-1}x_2 - a\sqrt{-1}x_3 - ax_4)^n (b^{-1}x_1 + b\sqrt{-1}x_2 + b^{-1}\sqrt{-1}x_3 - bx_4)^n \times e^{-\pi\left[(a^{-2}+b^{-2})x_1^2+(a^{-2}+b^2)x_2^2+(a^2+b^{-2})x_3^2+(a^2+b^2)x_4^2\right]}e^{2\pi\sqrt{-1}(z_1x_1+z_2x_2+z_3x_3+z_4x_4)}\,dx_1\,dx_2\,dx_3\,dx_4.$$

Then

$$f_0(z) = (ab+a^{-1}b^{-1})^{-1}(ab^{-1}+a^{-1}b)^{-1}e^{-\pi\left[(a^{-2}+b^{-2})^{-1}z_1^2+(a^{-2}+b^2)^{-1}z_2^2+(a^2+b^{-2})^{-1}z_3^2+(a^2+b^2)^{-1}z_4^2\right]}$$

and $f_n(z) = \nabla^n f_0(z)$, where

$$\nabla = \frac{1}{(2\pi\sqrt{-1})^2}\left(a^{-1}\frac{\partial}{\partial z_1} - a^{-1}\sqrt{-1}\frac{\partial}{\partial z_2} - a\sqrt{-1}\frac{\partial}{\partial z_3} - a\frac{\partial}{\partial z_4}\right) \times \left(b^{-1}\frac{\partial}{\partial z_1} + b\sqrt{-1}\frac{\partial}{\partial z_2} + b^{-1}\sqrt{-1}\frac{\partial}{\partial z_3} - b\frac{\partial}{\partial z_4}\right).$$

We claim that

$$f_n(0) = \pi^{-n}\Gamma(n+1)(ab+a^{-1}b^{-1})^{-1}(ab^{-1}+a^{-1}b)^{-1}\left[(ab+a^{-1}b^{-1})^{-1}+(ab^{-1}+a^{-1}b)^{-1}\right]^n. \tag{8.2}$$

Assume the claim holds. We have

$$\Phi([\mathbf{m}(a), \mathbf{m}(b)]) = \int_{V^2(\mathbb{R})} \varphi\left(\mathbf{m}(a)^{-1}x, \mathbf{m}(a)^{-1}y\right)\overline{\varphi(x\mathbf{m}(b)^{-1}, y\mathbf{m}(b)^{-1})}\,dx\,dy.$$

Making a change of variables from (y_1, y_3) to $(-y_1, -y_3)$, we see that

$$\begin{aligned}\Phi([\mathbf{m}(a), \mathbf{m}(b)]) &= f_{\lambda_1}(0)f_{-\lambda_2}(0)\\ &= \pi^{-\kappa_1}\Gamma(\lambda_1+1)\Gamma(-\lambda_2+1)(ab+a^{-1}b^{-1})^{-2}(ab^{-1}+a^{-1}b)^{-2}\\ &\quad\times\left[(ab+a^{-1}b^{-1})^{-1}+(ab^{-1}+a^{-1}b)^{-1}\right]^{\kappa_1}.\end{aligned}$$

Here the last equality follows from (8.2).

It remains to prove the claim (8.2). Let

$$D = \{(a,b) \in \mathbb{C}^2 \mid ab(ab+a^{-1}b^{-1})(ab^{-1}+a^{-1}b) \neq 0\}.$$

Let $(a, b) \in D$ be such that $(1+ab^{-1})(1+a^{-1}b^{-1})(1-a^{-1}b)(1-ab) \neq 0$. We make the following change of variables:

$$\begin{cases} z_1 = a^{-1}u_1 + b^{-1}u_4, \\ z_4 = -bu_1 + au_4, \end{cases} \qquad \begin{cases} z_2 = a^{-1}u_2 + bu_3, \\ z_3 = -b^{-1}u_2 + au_3, \end{cases}$$

$$\begin{cases} Z = (1+ab^{-1})^{-1}u_1 + (1+a^{-1}b^{-1})^{-1}\sqrt{-1}u_3, \\ \overline{Z} = (1+ab^{-1})^{-1}u_1 - (1+a^{-1}b^{-1})^{-1}\sqrt{-1}u_3, \end{cases} \qquad \begin{cases} Z' = (1-a^{-1}b)^{-1}u_4 + (1-ab)^{-1}\sqrt{-1}u_2, \\ \overline{Z'} = (1-a^{-1}b)^{-1}u_4 - (1-ab)^{-1}\sqrt{-1}u_2, \end{cases}$$

$$\begin{cases} Z = u + u', \\ \overline{Z'} = -u + u', \end{cases} \qquad \begin{cases} \overline{Z} = v + v', \\ Z' = v - v'. \end{cases}$$

Then

$$\begin{aligned} f_0(z) &= (ab+a^{-1}b^{-1})^{-1}(ab^{-1}+a^{-1}b)^{-1}e^{-4\pi\left[(ab+a^{-1}b^{-1})^{-1}+(ab^{-1}+a^{-1}b)^{-1}\right](uv+u'v')}\\ &\quad\times e^{-4\pi\left[(ab^{-1}+a^{-1}b)^{-1}-(ab+a^{-1}b^{-1})^{-1}\right](uu'+vv')-4\pi(uv'+u'v)},\\ \nabla &= \frac{1}{(2\pi\sqrt{-1})^2}\cdot\frac{\partial^2}{\partial u\partial v}. \end{aligned}$$

It follows that

$$\begin{aligned} f_n(0) &= (-4\pi^2)^{-n}(n!)^2(ab+a^{-1}b^{-1})^{-1}(ab^{-1}+a^{-1}b)^{-1} \\ &\quad \times \Big(\text{the coefficient of } u^nv^n \text{ in the Taylor expansion of } e^{-4\pi\left[(ab+a^{-1}b^{-1})^{-1}+(ab^{-1}+a^{-1}b)^{-1}\right]uv} \text{ at } (0,0)\Big) \\ &= \pi^{-n}\Gamma(n+1)(ab+a^{-1}b^{-1})^{-1}(ab^{-1}+a^{-1}b)^{-1}\left[(ab+a^{-1}b^{-1})^{-1}+(ab^{-1}+a^{-1}b)^{-1}\right]^n. \end{aligned}$$

Therefore (8.2) holds in this case. Since both sides of (8.2) are holomorphic functions on D, we conclude that they are equal on D. In particular, (8.2) holds for $a, b > 0$. This completes the proof. □

Lemma 8.7. *Let $v \in S(\mathrm{PS})$. For $a, b > 0$, we have*

$$\Phi_v([\mathbf{m}(a), \mathbf{m}(b)]) = (ab+a^{-1}b^{-1})^{-2}(ab^{-1}+a^{-1}b)^{-2}.$$

Proof. The calculation is similar to the one in Lemma 8.6. We leave the details to the readers. □

8.2. **Rallis inner product formula.** Let $\mathcal{B}_\sigma : \mathcal{V}_\sigma \otimes \overline{\mathcal{V}}_\sigma \to \mathbb{C}$ and $\mathcal{B}_{\sigma^\sharp} : \mathcal{V}_{\sigma^\sharp} \otimes \overline{\mathcal{V}}_{\sigma^\sharp} \to \mathbb{C}$ be the Petersson pairings defined by

$$\mathcal{B}_\sigma(f_1, f_2) = \int_{Z_H(\mathbb{A})H^\circ(k)\backslash H^\circ(\mathbb{A})} f_1(h_0)\overline{f_2(h_0)}\, dh_0,$$
$$\mathcal{B}_{\sigma^\sharp}(f_3, f_4) = \int_{Z_H(\mathbb{A})H(k)\backslash H(\mathbb{A})} f_3(h)\overline{f_4(h)}\, dh$$

for $f_1, f_2 \in \mathcal{V}_\sigma$ and $f_3, f_4 \in \mathcal{V}_{\sigma^\sharp}$. Here Z_H is the center of H, and dh_0 and dh are the Tamagawa measures on $Z_H(\mathbb{A})\backslash H^\circ(\mathbb{A})$ and $Z_H(\mathbb{A})\backslash H(\mathbb{A})$, respectively. Note that $\mathrm{vol}(Z_H(\mathbb{A})H^\circ(k)\backslash H^\circ(\mathbb{A})) = 4$ and $\mathrm{vol}(Z_H(\mathbb{A})H(k)\backslash H(\mathbb{A})) = 2$. By [Wal85, Proposition 6] and [GI11, Lemma 2.3], we have

$$\mathcal{B}_\sigma = \mathfrak{D}^{-1} \cdot \frac{4}{\xi(2)^2} \cdot L(1, \sigma, \mathrm{Ad}) \cdot \prod_v \mathcal{B}_v, \tag{8.3}$$

$$\mathcal{B}_{\sigma^\sharp} = \mathfrak{D}^{-1} \cdot \frac{4}{\xi(2)^2} \cdot L(1, \sigma, \mathrm{Ad}) \cdot \prod_v \mathcal{B}_v^\sharp. \tag{8.4}$$

Here $\mathcal{B}_v$ and $\mathcal{B}_v^\sharp$ are the local Hermitian pairings defined in § 8.1.1.

Let $\mathcal{B}_\omega : \mathcal{S}(V^2(\mathbb{A})) \otimes \overline{\mathcal{S}(V^2(\mathbb{A}))} \to \mathbb{C}$ be the canonical pairing defined by

$$\mathcal{B}_\omega(\varphi_1, \varphi_2) = \int_{V^2(\mathbb{A})} \varphi_1(x)\overline{\varphi_2(x)}\, dx,$$

where dx is the Tamagawa measure on $V^2(\mathbb{A})$. For each place v of k, let $\mathcal{B}_{\omega_v} : \mathcal{S}(V^2(k_v)) \otimes \overline{\mathcal{S}(V^2(k_v))} \to \mathbb{C}$ be the canonical pairing defined by

$$\mathcal{B}_{\omega_v}(\varphi_1, \varphi_2) = \int_{V^2(k_v)} \varphi_1(x_v)\overline{\varphi_2(x_v)}\, dx_v,$$

where dx_v is defined by the Haar measure on k_v in § 2.2. Then we have

$$\mathcal{B}_\omega(\varphi_1, \varphi_2) = \mathfrak{D}^{-4} \cdot \prod_v \mathcal{B}_{\omega_v}(\varphi_{1,v}, \varphi_{2,v}) \tag{8.5}$$

for $\varphi_1 = \bigotimes_v \varphi_{1,v}$, $\varphi_2 = \bigotimes_v \varphi_{2,v} \in \mathcal{S}(V^2(\mathbb{A}))$.

Theorem 8.8 (Rallis inner product formula). *Let $f = \bigotimes_v f_v \in \sigma^\sharp$ and $\varphi = \bigotimes_v \varphi_v \in \mathcal{S}(V^2(\mathbb{A}))$. We have*

$$\langle \theta(f, \varphi), \theta(f, \varphi) \rangle = \mathfrak{D}^{-8} \cdot \frac{4}{\xi(2)^4} \cdot \frac{L(1, \pi, \mathrm{Ad})}{\Delta_{\mathrm{PGSp}_4}} \cdot \prod_v Z_v(f_v, \varphi_v),$$

where

$$Z_v(f_v, \varphi_v) = \frac{\zeta_v(2)\zeta_v(4)}{L(1, \sigma_v, \mathrm{std})} \cdot \int_{H_1(k_v)} \mathcal{B}_{\omega_v}(\omega_v(1, h_{1,v})\varphi_v, \varphi_v)\mathcal{B}_v^\sharp(\sigma_v^\sharp(h_{1,v})f_v, f_v)\, dh_{1,v}.$$

Proof. Since

$$L(s,\pi,\mathrm{Ad}) = L(s,\sigma,\mathrm{std}) \cdot L(s,\sigma,\mathrm{Ad}),$$

the assertion is a special case of the Rallis inner product formula proved in [GQT14, Theorem 11.3] (see also [GI11, §7]). Here the extra factor $4\mathfrak{D}^{-8}\xi(2)^{-4}L(1,\sigma,\mathrm{Ad})$ is due to the ratio of the Haar measures in (7.4), (8.4), and (8.5). □

8.3. **Calculation of local integrals.** Let v be a place of k. Let $\mathbf{f}_v^\sharp \in \mathcal{V}_v^\sharp$ be the local component of $\mathbf{f}^\sharp$ with respect to the isomorphism in (7.6), and $\varphi_v \in S(V^2(k_v))$ the Schwartz function defined in (6.2)-(6.5). Let Ψ_v and Φ_v be the matrix coefficients of $\sigma_v^\sharp$ and ω_v, respectively, defined by

$$\Psi_v(h_v) = \mathcal{B}_v^\sharp(\sigma_v^\sharp(h_v)\mathbf{f}_v^\sharp, \mathbf{f}_v^\sharp),$$
$$\Phi_v(h_{1,v}) = \mathcal{B}_{\omega_v}(\omega_v(1,h_{1,v})\varphi_v, \varphi_v)$$

for $h_v \in H(k_v)$ and $h_{1,v} \in H_1(k_v)$. Let Z_v be the local integral defined by

$$Z_v = \frac{\zeta_v(2)\zeta_v(4)}{L(1,\sigma_v,\mathrm{std})} \cdot \int_{H_1(k_v)} \Phi_v(h_{1,v})\Psi_v(h_{1,v})\, dh_{1,v}.$$

Lemma 8.9. *Let v be a finite place of k. If $v \nmid \mathfrak{n}$, then*

$$Z_v = 1.$$

If $v \mid \mathfrak{n}$, then

$$Z_v = 2^{-2} \cdot q_v^{-3} \frac{\zeta_v(2)\zeta_v(4)}{\zeta_v(1)^2}.$$

Proof. We drop the subscript v for brevity. First we assume $v \nmid \mathfrak{n}$. By Lemmas 8.1, 8.5, (7.9), and [PSR87, Proposition 6.2] (see also [LR05, Proposition 3], [HN18, Proposition 6.3]), we have

$$\int_{H_1(k_v)} \Phi(h_1)\Psi(h_1)\, dh_1 = \frac{L(1,\sigma,\mathrm{std})}{\zeta(2)\zeta(4)}.$$

Now we assume $v \mid \mathfrak{n}$. The calculations for $v \mid \mathfrak{n}_1$ and $v \mid \mathfrak{n}_2$ are similar and we assume $v \mid \mathfrak{n}_1$. Since $v \in \mathfrak{S}$ and $\mathbf{f}^\sharp = (W,0) \in \mathcal{V}^\sharp$, we have

$$\Psi(h) = 2^{-1}\mathcal{B}(\sigma(h)W,W),$$
$$\Psi(h\mathbf{t}) = \mathcal{B}^\sharp((0,\sigma(\mathrm{Ad}(\mathbf{t})h)W),(W,0)) = 0$$

for $h \in H^\circ(k_v)$. Therefore

$$Z = 2^{-2} \cdot \frac{\zeta(2)\zeta(4)}{L(1,\sigma,\mathrm{std})} \cdot \int_{H_1^\circ(k_v)} \Phi(h_1)\mathcal{B}(\sigma(h_1)W,W)\, dh_1.$$

Let $\mathcal{U}$ be the open compact subgroup of $H_1^\circ(k_v)$ defined by

$$\mathcal{U} = H_1^\circ(k_v) \cap (K_0(\varpi) \times \mathrm{GL}_2(\mathfrak{o}))/\mathfrak{o}^\times.$$

For $n,m \in \mathbb{Z}$, let

$$h_{n,m} = [\mathbf{a}(\varpi^n)\mathbf{d}(\varpi^m), \mathbf{a}(\varpi^{n+m})], \quad h'_{n,m} = [w\mathbf{a}(\varpi^n)\mathbf{d}(\varpi^m), \mathbf{a}(\varpi^{n+m})]$$

with the notation of § 2.1. Note that

$$\left\{h_{n,m}, h'_{n,m} \mid n,m \in \mathbb{Z},\, n+m \geq 0\right\}$$

is a complete set of coset representatives for $\mathcal{U}\backslash H_1^\circ(k_v)/\mathcal{U}$, and we have

$$|\,\mathcal{U}h_{n,m}\mathcal{U}/\mathcal{U}\,| = \begin{cases} q^{|n-m|} & \text{if } n+m=0, \\ q^{n+m+|n-m|}(1+q^{-1}) & \text{if } n+m \geq 1, \end{cases}$$
$$(8.6)\qquad |\,\mathcal{U}h'_{n,m}\mathcal{U}/\mathcal{U}\,| = \begin{cases} q^{|n-m-1|} & \text{if } n+m=0, \\ q^{n+m+|n-m-1|}(1+q^{-1}) & \text{if } n+m \geq 1. \end{cases}$$

By Lemmas 8.2, 8.5, (7.10), and (8.6), we have

$$\begin{aligned}
\mathrm{vol}(\mathcal{U}, dh_1)^{-1} \int_{H_1^\circ(k_v)} \Phi(h_1)\mathcal{B}(\sigma(h_1)W, W)\, dh_1 &= \sum_{n,m\in\mathbb{Z},\, n+m\geq 0} \Phi(h_{n,m})\mathcal{B}(\sigma(h_{n,m})W, W) \cdot |\,\mathcal{U}h_{n,m}\mathcal{U}/\mathcal{U}\,| \\
&+ \sum_{n,m\in\mathbb{Z},\, n+m\geq 0} \Phi(h'_{n,m})\mathcal{B}(\sigma(h'_{n,m})W, W) \cdot |\,\mathcal{U}h'_{n,m}\mathcal{U}/\mathcal{U}\,| \\
&= \frac{\zeta(2)}{\zeta(1)} \cdot (Z^{(1)} + Z^{(2)} + Z^{(3)} + Z^{(4)}),
\end{aligned}$$

where

$$\begin{aligned}
Z^{(1)} &= \sum_{n\in\mathbb{Z}} \left(q^{-4|n|-2} - q^{-2|n|-2|n-1|-2} \right), \\
Z^{(2)} &= (q^{-2} - q^{-4}) \sum_{m=1}^{\infty} \varepsilon^m q^{-3m/2} \left(\alpha^m \cdot \frac{1-\alpha^{-2}q^{-1}}{1-\alpha^{-2}} + \alpha^{-m} \cdot \frac{1-\alpha^2 q^{-1}}{1-\alpha^2} \right), \\
Z^{(3)} &= (q^{-2} - q^{-4}) \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \varepsilon^{-n-m} q^{-5n/2-3m/2} \left(\alpha^{-n+m} \cdot \frac{1-\alpha^{-2}q^{-1}}{1-\alpha^{-2}} + \alpha^{n-m} \cdot \frac{1-\alpha^2 q^{-1}}{1-\alpha^2} \right), \\
Z^{(4)} &= (q^{-2} - 1) \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \varepsilon^{n+m} q^{-3n/2-5m/2} \left(\alpha^{n-m} \cdot \frac{1-\alpha^{-2}q^{-1}}{1-\alpha^{-2}} + \alpha^{-n+m} \cdot \frac{1-\alpha^2 q^{-1}}{1-\alpha^2} \right),
\end{aligned}$$

with α and ε as in (8.1). By direct calculations, we have

$$\begin{aligned}
Z^{(1)} &= \frac{(q-q^{-1})^2}{q^4-1}, \\
Z^{(2)} &= (q^{-2} - q^{-4}) \left[\varepsilon q^{-3/2}(\alpha + \alpha^{-1}) - q^{-3} - q^{-4} \right] (1 - \varepsilon\alpha q^{-3/2})^{-1} (1 - \varepsilon\alpha^{-1} q^{-3/2})^{-1}, \\
Z^{(3)} &= \frac{q^{-2} - q^{-4}}{q^4-1} \cdot \left[\varepsilon q^{-3/2}(\alpha + \alpha^{-1}) - q^{-3} - q^{-4} \right] (1 - \varepsilon\alpha q^{-3/2})^{-1} (1 - \varepsilon\alpha^{-1} q^{-3/2})^{-1}, \\
Z^{(4)} &= \frac{q^{-2} - 1}{q^4-1} \cdot \left[\varepsilon q^{-3/2}(\alpha + \alpha^{-1}) - q^{-3} - q^{-4} \right] (1 - \varepsilon\alpha q^{-3/2})^{-1} (1 - \varepsilon\alpha^{-1} q^{-3/2})^{-1}.
\end{aligned}$$

Therefore, noting that $\mathrm{vol}(\mathcal{U}, dh_1) = (1+q)^{-1}$ by the normalization (7.1) and that

$$L(s, \sigma, \mathrm{std}) = (1 - \varepsilon\alpha q^{-s-1/2})^{-1}(1 - \varepsilon\alpha^{-1} q^{-s-1/2})^{-1},$$

we have

$$\begin{aligned}
\int_{H_1^\circ(k_v)} \Phi(h_1)\mathcal{B}(\sigma(h_1)W, W) dh_1 &= q^{-3}(1-q^{-1})^2 (1 - \varepsilon\alpha q^{-3/2})^{-1}(1 - \varepsilon\alpha^{-1} q^{-3/2})^{-1} \\
&= \frac{L(1, \sigma, \mathrm{std})}{\zeta(2)\zeta(4)} \cdot q^{-3} \frac{\zeta(2)\zeta(4)}{\zeta(1)^2}.
\end{aligned}$$

This completes the proof. □

Lemma 8.10. *Let $v \in S(\mathrm{DS})$. We have*

$$Z_v = 2^{-\lambda_{1,v}-\lambda_{2,v}-3}(1 + \lambda_{1,v} - \lambda_{2,v})^{-1} \cdot \begin{cases} 1 & \textit{if } v \notin \mathfrak{S}, \\ 2^{-2} & \textit{if } v \in \mathfrak{S}. \end{cases}$$

Proof. We identify $k_v = \mathbb{R}$ and drop the subscript v for brevity. Without loss of generality, we assume $\kappa_1 \geq \kappa_2$. If $v \notin \mathfrak{S}$, then $\kappa_1 = \kappa_2$ and we have $\omega(1, \mathbf{t})\varphi = (-1)^{\lambda_1}\varphi = \varphi$, $\sigma^\sharp(\mathbf{t})W = W$, and $\Psi(h) = \mathcal{B}(\sigma(h)W, W)$ for $h \in H^\circ(\mathbb{R})$. If $v \in \mathfrak{S}$, then

$$\begin{aligned}
\Psi(h) &= 2^{-1}\mathcal{B}(\sigma(h)W, W), \\
\Psi(h\mathbf{t}) &= \mathcal{B}^\sharp((0, \sigma(\mathrm{Ad}(\mathbf{t})h)W), (W, 0)) = 0
\end{aligned}$$

for $h \in H^\circ(\mathbb{R})$. Therefore, we have

$$Z = \frac{\zeta(2)\zeta(4)}{L(1,\sigma,\mathrm{std})} \cdot \int_{H_1^\circ(\mathbb{R})} \Phi(h_1)\mathcal{B}(\sigma(h_1)W,W)\,dh_1 \cdot \begin{cases} 1 & \text{if } v \notin \mathfrak{S}, \\ 2^{-2} & \text{if } v \in \mathfrak{S}. \end{cases}$$

By (7.3), (7.12), and Lemmas 8.3, 8.6, we have

$$\begin{aligned}
&\int_{H_1^\circ(\mathbb{R})} \Phi(h_1)\mathcal{B}(\sigma(h_1)W,W)\,dh_1 \\
&= 2^{-2-2\kappa_1-\kappa_2}\pi^{2-\kappa_1}\Gamma(\lambda_1+1)\Gamma(-\lambda_2+1) \\
&\times \int_0^\infty \int_0^\infty \cosh(t_1+t_2)^{-2}\cosh(t_1-t_2)^{-2}\left(\cosh(t_1+t_2)^{-1}+\cosh(t_1-t_2)^{-1}\right)^{\kappa_1} \\
&\qquad\qquad \times \cosh(t_1)^{-\kappa_1}\cosh(t_2)^{-\kappa_2}\sinh(2t_1)\sinh(2t_2)\,dt_1\,dt_2 \\
&= 2^{-\kappa_2+1}\pi^{2-\kappa_1}\Gamma(\lambda_1+1)\Gamma(-\lambda_2+1) \\
&\times \int_0^\infty \int_0^\infty \frac{\cosh(t_2)^{-2\lambda_2+1}\sinh(2t_1)\sinh(t_2)}{(\cosh(2t_1)+\cosh(2t_2))^{2+\kappa_1}}\,dt_1\,dt_2.
\end{aligned}$$

Here the last equality follows from the formulae

$$\begin{aligned}
\cosh(t_1+t_2)\cosh(t_1-t_2) &= 2^{-1}(\cosh(2t_1)+\cosh(2t_2)), \\
\cosh(t_1+t_2)+\cosh(t_1-t_2) &= 2\cosh(t_1)\cosh(t_2), \\
\sinh(2t_2) &= 2\sinh(t_2)\cosh(t_2).
\end{aligned}$$

Moreover, the above integral is equal to

$$\begin{aligned}
&\int_0^\infty \cosh(t_2)^{-2\lambda_2+1}\sinh(t_2)\left(\int_0^\infty \frac{\sinh(2t_1)}{(\cosh(2t_1)+\cosh(2t_2))^{2+\kappa_1}}\,dt_1\right)dt_2 \\
&= 2^{-1}(1+\kappa_1)^{-1}\int_0^\infty \frac{\cosh(t_2)^{-2\lambda_2+1}\sinh(t_2)}{(1+\cosh(2t_2))^{1+\kappa_1}}\,dt_2 \\
&= 2^{-2-\kappa_1}(1+\kappa_1)^{-1}\int_0^\infty \cosh(t_2)^{-2\lambda_1-1}\sinh(t_2)\,dt_2 \\
&= 2^{-3-\kappa_1}\lambda_1^{-1}(1+\kappa_1)^{-1}.
\end{aligned}$$

Noting that

$$L(s,\sigma,\mathrm{std}) = 2^2(2\pi)^{-2s-\lambda_1+\lambda_2+1}\Gamma(s+\lambda_1-1)\Gamma(s-\lambda_2),$$

we conclude that

$$\begin{aligned}
\int_{H_1^\circ(\mathbb{R})} \Phi(h_1)\mathcal{B}(\sigma(h_1)W,W)\,dh_1 &= 2^{-2-2\lambda_1}(1+\kappa_1)^{-1}\pi^{2-\kappa_1}\Gamma(\lambda_1)\Gamma(-\lambda_2+1) \\
&= \frac{L(1,\,\sigma,\,\mathrm{std})}{\zeta(2)\zeta(4)}\cdot 2^{-\lambda_1-\lambda_2-3}(1+\kappa_1)^{-1}.
\end{aligned}$$

This completes the proof. □

Lemma 8.11. *Let $v \in S(\mathrm{PS})$. We have*

$$Z_v = 2^{-4}.$$

Proof. We identify $k_v = \mathbb{R}$ and drop the subscript v for brevity. We follow the doubling method in [PSR87, §6] and [LR05]. Let

$$\mathrm{O}_{2,2} = \left\{ h \in \mathrm{GL}_4 \;\middle|\; h\begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}{}^t h = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \right\}.$$

We identify $H_1(\mathbb{R})$ with $\mathrm{O}_{2,2}(\mathbb{R})$ with respect to the following basis of V:

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let Q be the standard Siegel parabolic subgroup of $\mathrm{O}_{2,2}$, N its unipotent radical, and $K_{2,2} = \mathrm{O}_{2,2}(\mathbb{R}) \cap \mathrm{O}(4)$ a maximal compact subgroup of $\mathrm{O}_{2,2}(\mathbb{R})$. We regard GL_2 as a Levi subgroup of Q via the embedding

$$\mathrm{GL}_2 \longrightarrow Q, \quad g \longmapsto \underline{g} = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}.$$

Then we have

$$\sigma^\sharp|_{\mathrm{O}_{2,2}(\mathbb{R})} = \mathrm{Ind}_{Q(\mathbb{R})}^{\mathrm{O}_{2,2}(\mathbb{R})}(\tau),$$

where

$$\tau = \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{\mu_1+\mu_2} \boxtimes |\ |^{\mu_1-\mu_2}).$$

Denote by ω_τ and $\omega_{\tau^\vee}$ the bi-O(2)-invariant matrix coefficients of τ and $\tau^\vee$, respectively, normalized so that $\omega_\tau(1) = \omega_{\tau^\vee}(1) = 1$. We identify $\mathrm{O}_{2,2} \times \mathrm{O}_{2,2}$ with its image under the embedding

$$\mathrm{O}_{2,2} \times \mathrm{O}_{2,2} \longrightarrow \mathrm{O}_{4,4}, \quad \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \longrightarrow \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & -b' \\ c & 0 & d & 0 \\ 0 & -c' & 0 & d' \end{pmatrix}.$$

Let P be the standard Siegel parabolic subgroup of $\mathrm{O}_{4,4}$ and $K_{4,4} = \mathrm{O}_{4,4}(\mathbb{R}) \cap \mathrm{O}(8)$ a maximal compact subgroup of $\mathrm{O}_{4,4}(\mathbb{R})$. We regard GL_4 as a Levi subgroup of P via the embedding

$$\mathrm{GL}_4 \longrightarrow P, \quad g \longmapsto \underline{g} = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}.$$

Let f^o be the right $K_{4,4}$-invariant section of the degenerate principal series representation $\mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{O}_{4,4}(\mathbb{R})}(\delta_P^{s/3})$ normalized so that $f^o(\delta, s) = 1$, where

$$\delta = \begin{pmatrix} 0 & 0 & \frac{1}{2}\mathbf{1}_2 & \frac{1}{2}\mathbf{1}_2 \\ \frac{1}{2}\mathbf{1}_2 & -\frac{1}{2}\mathbf{1}_2 & 0 & 0 \\ \mathbf{1}_2 & \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_2 & -\mathbf{1}_2 \end{pmatrix}.$$

Consider the integral

$$\mathcal{Z}(s) = \int_{\mathrm{O}_{2,2}(\mathbb{R})} f^o(\delta(h,1), s)\Psi(h)\, dh.$$

By [GI11, Lemma 7.7], the integral $\mathcal{Z}(s)$ is absolutely convergent at $s = 1/2$. By Lemma 8.7 and [PSR87, Proposition 6.4], we have

$$\Phi(h) = 2^{-4} \cdot f^o\left(\delta(h,1), \frac{1}{2}\right)$$

for $h \in \mathrm{O}_{2,2}(\mathbb{R})$. Therefore

$$Z = 2^{-4} \cdot \frac{\zeta(2)\zeta(4)}{L(1, \sigma, \mathrm{std})} \cdot \mathcal{Z}\left(\frac{1}{2}\right).$$

Let $\langle\ ,\ \rangle_\tau : \tau \times \tau^\vee \to \mathbb{C}$ be a non-zero invariant pairing. Let $\phi \in \mathrm{Ind}_{Q(\mathbb{R})}^{\mathrm{O}_{2,2}(\mathbb{R})}(\tau)$ and $\phi^\vee \in \mathrm{Ind}_{Q(\mathbb{R})}^{\mathrm{O}_{2,2}(\mathbb{R})}(\tau^\vee)$ be non-zero $K_{2,2}$-invariant sections. Then

$$\Psi(h) = \Psi(1) \cdot \int_{K_{2,2}} \frac{\langle \phi(kh), \phi^\vee(k) \rangle_\tau}{\langle \phi(1), \phi^\vee(1) \rangle_\tau}\, dk$$

for $h \in \mathrm{O}_{2,2}(\mathbb{R})$. Note that $\Psi(1) = 2^{-4}$ by Lemma 8.4. Therefore, we have

$$\begin{aligned}\mathcal{Z}(s) &= 2^{-4} \int_{\mathrm{O}_{2,2}(\mathbb{R})} f^o(\delta(h,1),s) \int_{K_{2,2}} \frac{\langle \phi(kh), \phi^\vee(k)\rangle_\tau}{\langle \phi(1), \phi^\vee(1)\rangle_\tau} \, dk \, dh \\ &= 2^{-4} \int_{K_{2,2}} \int_{\mathrm{O}_{2,2}(\mathbb{R})} f^o(\delta(h,k),s) \frac{\langle \phi(h), \phi^\vee(k)\rangle_\tau}{\langle \phi(1), \phi^\vee(1)\rangle_\tau} \, dh \, dk \\ &= 2^{-4} \int_{K_{2,2}\times K_{2,2}} \int_{\mathrm{GL}_2(\mathbb{R})} \int_{N(\mathbb{R})} \delta_Q(\underline{g})^{-1/2} f^o(\delta(u\underline{g}k_1, k_2), s) \omega_\tau(g) \, du \, dg \, dk_1 \, dk_2 \\ &= 2^{-4} \int_{\mathrm{GL}_2(\mathbb{R})} \int_{N(\mathbb{R})} \delta_Q(\underline{g})^{-1/2} f^o(\delta(u\underline{g}, 1), s) \omega_\tau(g) \, du \, dg.\end{aligned}$$

Let

$$A = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{1}_2 \end{pmatrix}.$$

Then the function

$$F_1^o(g,s) = |\det(g)|^{-1/2} \int_{N(\mathbb{R})} f^o(\delta(u,1)\underline{A}^{-1}\underline{g}, s) \, du$$

for $g \in \mathrm{GL}_4(\mathbb{R})$ defines an $\mathrm{O}(4)$-invariant section of $\mathrm{Ind}_{P'(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\delta_{P'}^{s/2})$, where

$$P' = \left\{ \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in \mathrm{GL}_2 \right\}.$$

On the other hand, an $\mathrm{O}(4)$-invariant section of $\mathrm{Ind}_{P'(\mathbb{R})}^{\mathrm{GL}_4(\mathbb{R})}(\delta_{P'}^{s/2})$ is also given by

$$F_2^o(g,s) = |\det(g)|^{s+1} \int_{\mathrm{GL}_2(\mathbb{R})} \Phi^o((0,x)g) |\det(x)|^{2s+2} \, dx$$

for $g \in \mathrm{GL}_4(\mathbb{R})$, where $\Phi^o \in \mathcal{S}(\mathrm{M}_{2,4}(\mathbb{R}))$ is defined by

$$\Phi^o(x) = e^{-\pi \, \mathrm{tr}(x\,{}^t x)}.$$

Therefore,

$$F_1^o = \frac{F_1^o(A,s)}{F_2^o(A,s)} \cdot F_2^o.$$

By direct calculations, we have

$$F_1^o(A,s) = 2 \cdot \frac{\zeta(2s+2)}{\zeta(2s+3)}, \quad F_2^o(A,s) = 2^{-2s-2}\zeta(2s+1)\zeta(2s+2).$$

Hence

$$\frac{F_1^o(A,s)}{F_2^o(A,s)} = 2^{2s+3}\zeta(2s+1)^{-1}\zeta(2s+3)^{-1}.$$

We conclude that

$$\begin{aligned}&\int_{\mathrm{GL}_2(\mathbb{R})} \int_{N(\mathbb{R})} \delta_Q(\underline{g})^{-1/2} f^o(\delta(u\underline{g},1),s) \omega_\tau(g) \, du \, dg \\ &= \int_{\mathrm{GL}_2(\mathbb{R})} F_1^o\left(A \begin{pmatrix} g & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}, s\right) \omega_\tau(g) \, dg \\ &= 2^{2s+3}\zeta(2s+1)^{-1}\zeta(2s+3)^{-1} \int_{\mathrm{GL}_2(\mathbb{R})} |\det(g_1)|^{s+1} \left(\int_{\mathrm{GL}_2(\mathbb{R})} \Phi^o(g_2 g_1, g_2) |\det(g_2)|^{2s+2} \, dg_2 \right) \omega_\tau(g_1) \, dg_1.\end{aligned}$$

Noting that ω_τ is a zonal spherical function on $\mathrm{GL}_2(\mathbb{R})$ with respect to $\mathrm{O}(2)$ and following the argument in the proof of [Mac71, Proposition (1.2.5)], we have

$$\int_{\mathrm{O}(2)} \omega_\tau(g_1 k g_2) \, dk = \omega_\tau(g_1)\omega_\tau(g_2)$$

for $g_1, g_2 \in \mathrm{GL}_2(\mathbb{R})$. Therefore, proceeding as in the proof of [PSR87, Proposition 6.1], we have

$$\begin{aligned}
&\int_{\mathrm{GL}_2(\mathbb{R})} |\det(g_1)|^{s+1} \left(\int_{\mathrm{GL}_2(\mathbb{R})} \Phi^o(g_2 g_1, g_2) |\det(g_2)|^{2s+2}\, dg_2 \right) \omega_\tau(g_1)\, dg_1 \\
&= \int_{\mathrm{GL}_2(\mathbb{R})} e^{-\pi\,\mathrm{tr}(g^t g)} \omega_\tau(g) |\det(g)|^{s+1}\, dg \int_{\mathrm{GL}_2(\mathbb{R})} e^{-\pi\,\mathrm{tr}(g^t g)} \omega_{\tau^\vee}(g) |\det(g)|^{s+1}\, dg \\
&= L\left(s + \frac{1}{2}, \sigma, \mathrm{std}\right).
\end{aligned}$$

Here the last equality follows from a calculation analogous to [GJ72, Lemma 6.10]. It follows that

$$\mathcal{Z}(s) = 2^{2s-1} \cdot \frac{L(s+1/2, \sigma, \mathrm{std})}{\zeta(2s+1)\zeta(2s+3)}.$$

This completes the proof. □

8.4. **Proof of Proposition 6.2.** The assertion follows from Theorem 8.8 and Lemmas 8.9-8.11.

9. Convergence lemmas

We keep the notation of § 5.2. In this section, we study the convergence of the doubling local zeta integrals and the Rankin-Selberg local zeta integrals defined in (4.4) and (4.5), respectively.

Except in Lemma 9.1, let π be a representation of $G(F)$ in one of the three types in § 5.2. In Case (IIa) and Case (PS), we fix a sign of representations and write $\pi = \pi_\lambda$ for the representation with parameter λ. Let Ω be the domain defined by excluding the points of reducibility of the induced representations (5.10) and (5.13) in Case (IIa) and Case (PS), respectively. Recall the domain

$$\mathcal{D} = \begin{cases} \{\lambda \in \mathbb{C} \mid |\mathrm{Re}(\lambda)| < 1/2\} & \text{in Case (IIa)}, \\ \left\{\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid |\mathrm{Re}(\lambda_1)| + |\mathrm{Re}(\lambda_2)| < 1\right\} & \text{in Case (PS)}, \end{cases}$$

defined in (5.15). Let

$$K = \begin{cases} G(\mathfrak{o}) & \text{if } F \text{ is non-archimedean}, \\ G(\mathbb{R}) \cap \mathrm{O}(4) & \text{if } F = \mathbb{R}. \end{cases}$$

9.1. **Doubling local zeta integrals.** Recall $P = P_{4,4}$ and $I(s) = I_{4,4}(s)$ in the notation of § 3.1. Denote by $\mathcal{C}(\pi)$ the space of matrix coefficients of π. If $\phi \in \mathcal{C}(\pi)$ and $F \in I(s)$ is a holomorphic section, let $Z(s, \phi, F)$ be the local zeta integal defined as in (4.4).

Lemma 9.1. *Assume π is an irreducible generic admissible unramified representation of $G(F)$ with trivial central character. Write*

$$\pi|_{\mathrm{Sp}_4(F)} = \mathrm{Ind}_{\mathrm{Sp}_4(F) \cap \mathbf{B}(F)}^{\mathrm{Sp}_4(F)}(|\ |^{\lambda_1} \boxtimes |\ |^{\lambda_2})$$

for some λ_1, $\lambda_2 \in \mathbb{C}$. Let $\phi \in \mathcal{C}(\pi)$ and let $F \in I(s)$ be a holomorphic section. The integral $Z(s, \phi, F)$ is absolutely convergent for $\mathrm{Re}(s) > -1/2 + \max\{|\mathrm{Re}(\lambda_1)|, |\mathrm{Re}(\lambda_2)|\}$.

Proof. Let $s \in \mathbb{R}$. Let $K_1 = \mathrm{Sp}_4(\mathfrak{o})$ and

$$A^+ = \{\mathrm{diag}(\varpi^{n_1}, \varpi^{n_2}, \varpi^{-n_1}, \varpi^{-n_2}) \,|\, n_1 \geq n_2 \geq 0\}.$$

Then $\mathrm{Sp}_4(F) = K_1 A^+ K_1$. We may assume that ϕ is bi-$G(\mathfrak{o})$-invariant, F is $\mathrm{GSp}_8(\mathfrak{o})$-invariant, and $F(1, s) = 1$. In particular,

$$F(\delta(k_1 g k_2, 1), s) = F(\delta(g, 1), s)$$

for all $k_1, k_2 \in K_1$. Let

$$a = \mathrm{diag}(\varpi^{n_1}, \varpi^{n_2}, \varpi^{-n_1}, \varpi^{-n_2}) \in A^+.$$

By [Wal03, p. 241], there exists a constant $C > 0$ such that

$$\mathrm{vol}(K_1 a K_1) \leq C q^{4n_1 + 2n_2}.$$

By [PSR87, Proposition 6.4],

$$F(\delta(a, 1), s) = q^{-(s+5/2)(n_1+n_2)}.$$

By Macdonald's formula [Mac71], [Cas80], there exists a polynomial Ψ such that

$$|\phi(a)| \leq \Psi(n_1) q^{-2n_1-n_2+\eta(n_1+n_2)},$$

where $\eta = \max\{|\mathrm{Re}(\lambda_1)|, |\mathrm{Re}(\lambda_2)|\}$. Hence $|Z(s,\phi,F)|$ is majorized by

$$\begin{aligned}&\sum_{a\in A^+} \mathrm{vol}(K_1aK_1)F(\delta(a,1),s)|\phi(a)|\\ &\leq C\sum_{n_2=0}^{\infty}\sum_{n_1=n_2}^{\infty} \Psi(n_1) q^{-(s+1/2-\eta)n_1} q^{-(s+3/2-\eta)n_2}.\end{aligned}$$

This completes the proof. □

Lemma 9.2. *Assume π is of type (DS). Let $\phi \in \mathcal{C}(\pi)$ and let $F \in I(s)$ be a holomorphic section. The integral $Z(s,\phi,F)$ is absolutely convergent for* $\mathrm{Re}(s) > -1/2$.

Proof. Fix $s > -1/2$. By Hölder's inequality, it suffices to show that the integral

$$\int_{\mathrm{Sp}_4(\mathbb{R})} |F(\delta(g,1),s)|^{2(1-\epsilon)}\, dg \tag{9.1}$$

is convergent for some $\epsilon > 0$. Let $K_1 = \mathrm{Sp}_4(\mathbb{R}) \cap \mathrm{O}(4)$ and

$$\mathfrak{a}^+ = \{\mathrm{diag}(X_1, X_2, -X_1, -X_2) \mid X_1 \geq X_2 \geq 0\}.$$

Then $\mathrm{Sp}_4(\mathbb{R}) = K_1 \exp(\mathfrak{a}^+) K_1$. We may assume that F is $(\mathrm{GSp}_8(\mathbb{R}) \cap \mathrm{O}(8))$-invariant and $F(1,s) = 1$. In particular,

$$F(\delta(k_1gk_2,1),s) = F(\delta(g,1),s)$$

for all $k_1, k_2 \in K_1$. Let Δ^+ be the set of positive roots determined by the chamber $\mathfrak{a}^+$. Let

$$X = \mathrm{diag}(X_1, X_2, -X_1, -X_2) \in \mathfrak{a}^+$$

and $a_i = \exp(X_i)$. By [PSR87, Proposition 6.4],

$$F(\delta(\exp(X),1),s) = \left(\prod_{i=1}^{2} \sqrt{(1+a_i^2)(1+a_i^{-2})}\right)^{-s-5/2} \leq (a_1a_2)^{-s-5/2}.$$

Obviously,

$$\left|\prod_{\alpha\in\Delta^+} \sinh(\alpha(X))\right| \leq a_1^4 a_2^2.$$

Hence the integral (9.1) is majorized by

$$\begin{aligned}&\int_{\mathfrak{a}^+}\int_{K_1\times K_1} |F(\delta(k_1\exp(X)k_2,1),s)|^{2(1-\epsilon)} \left|\prod_{\alpha\in\Delta^+}\sinh(\alpha(X))\right| dk_1\, dk_2\, dX\\ &\leq \int_1^{\infty}\int_{a_2}^{\infty} a_1^{-2s-1+2\epsilon s+5\epsilon} a_2^{-2s-3+2\epsilon s+5\epsilon}\, d^\times a_1\, d^\times a_2.\end{aligned}$$

This completes the proof. □

Assume $\pi = \pi_\lambda$ is of type (IIa) or (PS) with parameter λ. Recall that $P_{2,2}$ is the standard Siegel parabolic subgroup of G. We may write

$$\pi_\lambda = \begin{cases} \mathrm{Ind}_{P_{2,2}(F)}^{G(F)}(\tau_\lambda \boxtimes \eta^\varepsilon|\ |^{-\lambda}) & \text{in Case (IIa)},\\ \mathrm{Ind}_{P_{2,2}(\mathbb{R})}^{G(\mathbb{R})}(\tau_\lambda \boxtimes \mathrm{sgn}^\varepsilon|\ |^{(-\lambda_1-\lambda_2)/2}) & \text{in Case (PS)},\end{cases}$$

where τ_λ is the representation of $\mathrm{GL}_2(F)$ defined by

$$\tau_\lambda = \begin{cases} \mathrm{St}\otimes|\ |^\lambda & \text{in Case (IIa)},\\ \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{\lambda_1}\boxtimes|\ |^{\lambda_2}) & \text{in Case (PS)}.\end{cases}$$

Let

$$\mathbf{K} = \begin{cases} \mathrm{GL}_2(\mathfrak{o}) & \text{if } \pi \text{ is of type (IIa)}, \\ \mathrm{O}(2) & \text{if } \pi \text{ is of type (PS)}. \end{cases}$$

Note that the restrictions of π_λ and τ_λ to K and $\mathbf{K}$, respectively, do not depend on λ. We realize the representation τ_λ (resp. $\tau_\lambda^\vee$) on a space $\mathcal{V}$ (resp. $\mathcal{V}^\vee$) which does not depend on λ and on which the action of $\tau_\lambda|_{\mathbf{K}}$ (resp. $\tau_\lambda^\vee|_{\mathbf{K}}$) does not depend on λ, and fix a bilinear pairing $\langle\ ,\ \rangle : \mathcal{V} \times \mathcal{V}^\vee \to \mathbb{C}$ which is invariant with respect to the actions of τ_λ and $\tau_\lambda^\vee$ for all λ. We call a map

$$\Omega \longrightarrow C^\infty(G(F)), \quad \lambda \longmapsto \phi_\lambda$$

a K-finite analytic family of matrix coefficients if it satisfies the following conditions:

- The map $(\lambda, g) \mapsto \phi_\lambda(g)$ is continuous.
- For each $g \in G(F)$, the map $\lambda \mapsto \phi_\lambda(g)$ is analytic.
- For each $\lambda \in \Omega$, the function $g \mapsto \phi_\lambda(g)$ belongs to $\mathcal{C}(\pi_\lambda)$.
- There exist finitely many irreducible representations ρ_i of $K \times K$ such that ϕ_λ is contained in the direct sum of the ρ_i-isotypic subspaces of $\mathcal{C}(\pi_\lambda)$ for all $\lambda \in \Omega$.

We call a map

$$\Omega \times G(F) \longrightarrow \mathcal{V}, \quad (\lambda, g) \longmapsto h_\lambda(g)$$

a K-finite analytic section of π_λ if it satisfies the following conditions:

- For each $v \in \mathcal{V}^\vee$, the map $(\lambda, g) \mapsto \langle h_\lambda(g), v^\vee \rangle$ is continuous.
- For each $g \in G(F)$ and $v \in \mathcal{V}^\vee$, the map $\lambda \mapsto \langle h_\lambda(g), v^\vee \rangle$ is analytic.
- For each $\lambda \in \Omega$, the map $g \mapsto h_\lambda(g)$ belongs to π_λ.
- There exist finitely many irreducible representations ρ_i of K such that h_λ is contained in the direct sum of the ρ_i-isotypic subspaces of π_λ for all $\lambda \in \Omega$.

Similarly, we define the notion of K-finite analytic sections for $\pi_\lambda^\vee$. Given K-finite analytic sections h_λ and $h_\lambda^\vee$ of π_λ and $\pi_\lambda^\vee$, respectively, it is easy to show that the map

$$\lambda \longmapsto \left[g \mapsto \int_{K \cap \mathrm{Sp}_4(F)} \langle h_\lambda(kg), h_\lambda^\vee(k) \rangle \, dk \right] \tag{9.2}$$

defines a K-finite analytic family of matrix coefficients. Moreover, for any fixed $\lambda_0 \in \Omega$, any K-finite analytic family of matrix coefficients can be written as a linear combination, with analytic functions of λ as coefficients, of K-finite analytic families of the form (9.2) in a neighborhood of λ_0.

Lemma 9.3. *Let ϕ_λ be a K-finite analytic family of matrix coefficients and $F \in I(s)$ a holomorphic section. The integral $Z(s, \phi_\lambda, F)$ is absolutely convergent for*

$$\begin{cases} \mathrm{Re}(s) > -1/2 + \max\{0, |\mathrm{Re}(\lambda)| - 1/2\} & \text{in Case (IIa)}, \\ \mathrm{Re}(s) > -1/2 + \max\{|\mathrm{Re}(\lambda_1)|, |\mathrm{Re}(\lambda_2)|\} & \text{in Case (PS)}, \end{cases}$$

uniformly for λ varying in a compact set. In particular, the integral is absolutely convergent for $\mathrm{Re}(s) \geq 1/2$ *if $\lambda \in \mathcal{D}$.*

Proof. Since we only consider the convergence for λ varying in a compact set, we may assume ϕ_λ is of the form (9.2) for some K-finite analytic sections h_λ and $h_\lambda^\vee$ for π_λ and $\pi_\lambda^\vee$, respectively. Let Q be the standard Siegel parabolic subgroup of Sp_4 and N its unipotent radical. We identify GL_2 with a Levi subgroup of Q via the embedding

$$\mathrm{GL}_2 \longrightarrow Q, \quad g \longmapsto \underline{g} = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}.$$

Then

$$\begin{aligned} Z(s, \phi_\lambda, F) &= \int_{\mathrm{Sp}_4(F)} F(\delta(g,1), s) \int_{K \cap \mathrm{Sp}_4(F)} \langle h_\lambda(kg), h_\lambda^\vee(k) \rangle \, dk \, dg \\ &= \int_{K \cap \mathrm{Sp}_4(F)} \int_{\mathrm{Sp}_4(F)} F(\delta(g,k), s) \langle h_\lambda(g), h_\lambda^\vee(k) \rangle \, dg \, dk \\ &= \int_{(K \cap \mathrm{Sp}_4(F))^2} \int_{\mathrm{GL}_2(F)} \int_{N(F)} \delta_Q(\underline{g})^{-1/2} F(\delta(u\underline{g}k_1, k_2), s) \langle \tau_\lambda(g) h_\lambda(k_1), h_\lambda^\vee(k_2) \rangle \, du \, dg \, dk_1 \, dk_2. \end{aligned}$$

Recall that $P \cap \mathrm{Sp}_8$ is the standard Siegel parabolic subgroup of Sp_8. We identify GL_4 with a Levi subgroup of $P \cap \mathrm{Sp}_8$ via the embedding

$$\mathrm{GL}_4 \longrightarrow P \cap \mathrm{Sp}_8, \quad a \longmapsto \underline{a} = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}.$$

Let P' be the parabolic subgroup of GL_4 defined by

$$P' = \left\{ \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \;\middle|\; a, d \in \mathrm{GL}_2 \right\}.$$

For $g \in \mathrm{Sp}_8(F)$, define $\Psi(g,s) \in \mathrm{Ind}_{P'(F)}^{\mathrm{GL}_4(F)}(\delta_{P'}^{s/2})$ by

$$\Psi(g,s)(a) = |\det(a)|^{-3/2} \int_{N(F)} F(\delta(u,1)\underline{A}^{-1}\underline{a}\, g, s)\, du$$

for $a \in \mathrm{GL}_4(F)$. Here

$$A = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & -\mathbf{1}_2 \end{pmatrix}.$$

Note that $\Psi(g,s)$ is an intertwining integral (cf. [PSR87, Lemma 6.2] and [LR05, Proposition 1]) and absolutely convergent for $\mathrm{Re}(s) > -1/2$. Then

$$Z(s,\phi_\lambda,F) = \int_{(K\cap \mathrm{Sp}_4(F))^2} \int_{\mathrm{GL}_2(F)} \Psi((k_1,k_2),s) \left(A \begin{pmatrix} g & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} \right) \langle \tau_\lambda(g) h_\lambda(k_1), h_\lambda^\vee(k_2) \rangle \, dg\, dk_1\, dk_2.$$

Let $\eta = \max\{|\mathrm{Re}(\lambda_1)|, |\mathrm{Re}(\lambda_2)|\}$ and $\mu = \min\{\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2)\}$ in Case (PS). Fix $s \in \mathbb{R}$ such that

$$\begin{cases} s > -1/2 + \max\{0, |\mathrm{Re}(\lambda)| - 1/2\} & \text{in Case (IIa)}, \\ s > -1/2 + \eta & \text{in Case (PS)}. \end{cases}$$

It suffices to show that the integral

$$(9.3) \qquad \int_{(K\cap \mathrm{Sp}_4(F))^2} \int_{\mathrm{GL}_2(F)} \left| \Psi((k_1,k_2),s) \left(A \begin{pmatrix} g & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} \right) \langle \tau_\lambda(g) h_\lambda(k_1), h_\lambda^\vee(k_2) \rangle \right| dg\, dk_1\, dk_2$$

is uniformly convergent for λ varying in a compact set. We may assume the section F is $\mathrm{GSp}_8(\mathfrak{o})$-invariant (resp. $(\mathrm{GSp}_8(\mathbb{R}) \cap \mathrm{O}(8))$-invariant) in Case (IIa) (resp. Case (PS)). Let $f^o \in \mathrm{Ind}_{P'(F)}^{\mathrm{GL}_4(F)}(\delta_{P'}^{s/2})$ be the $\mathrm{GL}_4(\mathfrak{o})$-invariant section (resp. $\mathrm{O}(4)$-invariant section) in Case (IIa) (resp. Case (PS)) defined by

$$f^o(a,s) = |\det(a)|^{s+1} \int_{\mathrm{GL}_2(F)} \Phi^o((0,x)a) |\det(x)|^{2s+2}\, dx$$

for $a \in \mathrm{GL}_4(F)$. Here $\Phi^o \in \mathcal{S}(\mathrm{M}_{2,4}(F))$ is given by

$$\Phi^o(x) = \begin{cases} \mathbb{I}_{\mathrm{M}_{2,4}(\mathfrak{o})}(x) & \text{in Case (IIa)}, \\ e^{-\pi\, \mathrm{tr}(x\, {}^t x)} & \text{in Case (PS)}. \end{cases}$$

Note that the above integral is absolutely convergent for $\mathrm{Re}(s) > -1/2$. Then

$$\Psi(1,s) = \frac{\Psi(1,s)(A)}{f^o(A,s)} \cdot f^o.$$

Put $F^+ = \{\nu \in F^\times \mid |\nu| \leq 1\}$. There is a function κ on F^+ such that $0 \leq \kappa(\nu) \leq C \cdot |\nu|^{-1}$ for some constant C and such that

$$\int_{\mathrm{GL}_2(F)} f(g)\, dg = \int_{F^\times} \int_{F^+} \int_{\mathbf{K}\times\mathbf{K}} f(tk_1 \mathbf{a}(\nu) k_2) \kappa(\nu)\, dk_1\, dk_2\, d^\times \nu\, d^\times t$$

for all $f \in L^1(\mathrm{GL}_2(F))$. Let $0 < \epsilon < s + 1/2 - \eta$ in Case (PS). There exists a constant $C_\lambda > 0$ in Case (IIa) (resp. $C_{\lambda,\epsilon} > 0$ in Case (PS)) bounded uniformly as λ varies in a compact set such that

$$|\langle \tau_\lambda(k_1 \mathbf{a}(\nu) k_2) h_\lambda(k_3), h_\lambda^\vee(k_4) \rangle| \leq \begin{cases} C_\lambda \cdot |\nu|^{\mathrm{Re}(\lambda)+1} & \text{in Case (IIa)}, \\ C_{\lambda,\epsilon} \cdot |\nu|^{\mu+1/2-\epsilon} & \text{in Case (PS)}, \end{cases}$$

for all $\nu \in F^+$, $k_1, k_2 \in \mathbf{K}$, and $k_3, k_4 \in K$. In Case (IIa), we have

$$f^o\left(A\begin{pmatrix} \varpi^m \mathbf{a}(\varpi^n) & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}\right) = \zeta(2s+1)\zeta(2s+2)q^{(-n-2m)(s+1)} \cdot \begin{cases} 1 & \text{if } m \geq 0, \\ q^{m(2s+2)} & \text{if } 0 \geq m \geq -n, \\ q^{(n+2m)(2s+2)} & \text{if } -n \geq m, \end{cases}$$

for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. In Case (PS), we have

$$f^o\left(A\begin{pmatrix} t\mathbf{a}(\nu) & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}\right) = \zeta(2s+1)\zeta(2s+2)|\nu t^2|^{s+1}(1+\nu^2 t^2)^{-s-1}(1+t^2)^{-s-1}$$

for $t \in \mathbb{R}^\times$ and $\nu \in \mathbb{R}^+$. By [PSR87, Lemma 6.1],

$$f^o\left(A\begin{pmatrix} k_1 g k_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}\right) = f^o\left(A\begin{pmatrix} g & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}\right)$$

for all $k_1, k_2 \in \mathbf{K}$ and $g \in \mathrm{GL}_2(F)$. Therefore, the integral (9.3) is majorized by

$$\frac{\Psi(1,s)(A)}{f^o(A,s)} \cdot \zeta(2s+1)\zeta(2s+2)C \cdot \begin{cases} C_\lambda \cdot (I_\lambda^{(1)} + I_\lambda^{(2)} + I_\lambda^{(3)}) & \text{in Case (IIa)}, \\ C_{\lambda,\epsilon} \cdot I_{\lambda,\epsilon} & \text{in Case (PS)}, \end{cases}$$

where

$$\begin{aligned} I_\lambda^{(1)} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^{-n(s+\mathrm{Re}(\lambda)+1)} q^{-m(2s+2+2\mathrm{Re}(\lambda))}, \\ I_\lambda^{(2)} &= \sum_{n=0}^{\infty}\sum_{m=1}^{n} q^{-n(s+\mathrm{Re}(\lambda)+1)} q^{2m\mathrm{Re}(\lambda)}, \\ I_\lambda^{(3)} &= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} q^{-n(-s+\mathrm{Re}(\lambda)-1)} q^{-m(2s+2-2\mathrm{Re}(\lambda))}, \end{aligned}$$

in Case (IIa), and

$$I_{\lambda,\epsilon} = \int_{\mathbb{R}^\times}\left(\frac{t^2}{1+t^2}\right)^{s+1} |t|^{-s+\mathrm{Re}(\lambda_1)+\mathrm{Re}(\lambda_2)-\mu-1/2+\epsilon}\int_{-|t|}^{|t|}\frac{|\nu|^{s+\mu+1/2-\epsilon}}{(1+\nu^2)^{s+1}}\,d^\times\nu\,d^\times t$$

in Case (PS). This shows the absolute convergence of $Z(s,\phi_\lambda,F)$. It is clear that the integrals $I_\lambda^{(1)}, I_\lambda^{(2)}, I_\lambda^{(3)}, I_{\lambda,\epsilon}$ are bounded uniformly as λ varies in a compact set. This completes the proof. ☐

9.2. **Local zeta integrals for** $\mathrm{GSp}_4 \times \mathrm{GSp}_4$**.** Recall $\mathcal{P} = P_{4,3}$ and $\mathcal{I}(s) = I_{4,3}(s)$ in the notation of § 3.1. Denote by $C^\infty(U(F)\backslash G(F), \psi_U)$ the space of smooth Whittaker functions on $G(F)$ with respect to ψ_U and by $\mathcal{W}(\pi,\psi_U)$ the space of smooth Whittaker functions of π with respect to ψ_U. If $W \in \mathcal{W}(\pi,\psi_U)$ and $\mathcal{F} \in \mathcal{I}(s)$ is a holomorphic section, let $\mathcal{Z}(s,W,\mathcal{F})$ be the local zeta integal defined as in (4.5) with $\overline{W}$ replaced by the left translation of W by $\mathrm{diag}(-1,1,1,-1)$ when π is of type (IIa) or (PS).

We follow the notation in § 6.1. In particular, $V = \mathrm{M}_{2,2}$ is a quadratic space over F with the norm form. Let ω be the Weil representation of $\mathrm{Sp}_4(F) \times H_1(F)$ on $\mathcal{S}(V^2(F))$. We extend ω to a representation of $\mathrm{G}(\mathrm{Sp}_4 \times H_1)(F)$ as in (3.1). Let σ_1 and σ_2 be the irreducible admissible representations of $\mathrm{GL}_2(F)$ defined as follows:

- If π is of type (IIa), then

$$\sigma_1 = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(|\ |^\lambda \eta^\varepsilon \boxtimes |\ |^{-\lambda}\eta^\varepsilon), \quad \sigma_2 = \mathrm{St} \otimes \eta^\varepsilon.$$

- If π is of type (DS), then

$$\sigma_1 = \mathrm{DS}(\lambda_1 - \lambda_2), \quad \sigma_2 = \mathrm{DS}(\lambda_1 + \lambda_2).$$

- If π is of type (PS), then

$$\sigma_1 = \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{(\lambda_1+\lambda_2)/2}\mathrm{sgn}^\varepsilon \boxtimes |\ |^{(-\lambda_1-\lambda_2)/2}\mathrm{sgn}^\varepsilon), \quad \sigma_2 = \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(|\ |^{(\lambda_1-\lambda_2)/2}\mathrm{sgn}^\varepsilon \boxtimes |\ |^{(-\lambda_1+\lambda_2)/2}\mathrm{sgn}^\varepsilon).$$

Via the isomorphism

$$H^\circ \simeq \Delta\mathbb{G}_m\backslash(\mathrm{GL}_2 \times \mathrm{GL}_2),$$

we have a representation $\sigma = \sigma_1 \times \sigma_2$ of $H^\circ(F)$. Let $N^\square$ be the unipotent subgroup of H° defined by

$$N^\square = \{[\mathbf{n}(x), \mathbf{n}(y)] \mid x, y \in \mathbb{G}_a\},$$

and $\psi_{N^\square}$ the additive character of $N^\square(F)$ defined by

$$\psi_{N^\square}([\mathbf{n}(x), \mathbf{n}(y)]) = \psi(x + y).$$

Denote by $\mathcal{W}(\sigma, \psi_{N^\square})$ the space of smooth Whittaker functions of σ with respect to $\psi_{N^\square}$. We have a surjective equivariant map

$$S(V^2(F)) \otimes \mathcal{W}(\sigma, \psi_{N^\square}) \longrightarrow \mathcal{W}(\pi, \psi_U), \quad \varphi \otimes W \longmapsto \mathcal{W}(\varphi, W),$$

where

$$\mathcal{W}(\varphi, W)(g) = \int_{\Delta N(F)\backslash H_1^\circ(F)} W(h_1 h)\omega(g, h_1 h)\varphi(\mathbf{x}_0, \mathbf{y}_0)\, d\bar{h}_1 \tag{9.4}$$

for $(g, h) \in \mathrm{G}(\mathrm{Sp}_4 \times H_1)(F)$. Here

$$\mathbf{x}_0 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{y}_0 = \mathbf{a}(-1).$$

In Case (IIa) and Case (PS), we write $\pi = \pi_\lambda$ for the representation of $G(F)$ with parameter λ and $\sigma = \sigma_\lambda$ for the representation $H^\circ(F)$ defined as above with respect to λ. We call a map

$$\Omega \longrightarrow C^\infty(U(F)\backslash G(F), \psi_U), \quad \lambda \longmapsto W_\lambda$$

a K-finite analytic family of Whittaker functions if it satisfies the following conditions:

- The map $(\lambda, g) \mapsto W_\lambda(g)$ is continuous.
- For each $g \in G(F)$, the map $\lambda \mapsto W_\lambda(g)$ is analytic.
- For each $\lambda \in \Omega$, the function $g \mapsto W_\lambda(g)$ belongs to $\mathcal{W}(\pi_\lambda, \psi_U)$.
- There exist finitely many irreducible representations ρ_i of K such that W_λ is contained in the direct sum of the ρ_i-isotypic subspaces of $\mathcal{W}(\pi_\lambda, \psi_U)$ for all $\lambda \in \Omega$.

Let

$$\mathbf{K}^\square = \begin{cases} H^\circ(\mathfrak{o}) & \text{if } F \text{ is non-archimedean,} \\ (\mathrm{O}(2) \times \mathrm{O}(2))/\{\pm 1\} & \text{if } F = \mathbb{R}. \end{cases}$$

Similarly, we define the notion of $\mathbf{K}^\square$-finite analytic families valued in $\mathcal{W}(\sigma_\lambda, \psi_{N^\square})$.

Let $\epsilon > 0$. If π is of type (IIa), let $\mathfrak{X}(\pi, \epsilon)$ be the set consisting of the following characters of $\mathbf{T}(F)$:

$$\begin{aligned} &\mathrm{diag}(a, b, ca^{-1}, cb^{-1}) \longmapsto 1, \\ &\mathrm{diag}(a, b, ca^{-1}, cb^{-1}) \longmapsto |a|^{1/2} \cdot |b|^{1/2} \cdot |c|^{-1/2}, \\ &\mathrm{diag}(a, b, ca^{-1}, cb^{-1}) \longmapsto |a|^{-|\mathrm{Re}(\lambda)|-\epsilon} \cdot |b|^{-|\mathrm{Re}(\lambda)|-\epsilon} \cdot |c|^{|\mathrm{Re}(\lambda)|+\epsilon}, \\ &\mathrm{diag}(a, b, ca^{-1}, cb^{-1}) \longmapsto |a|^{-|\mathrm{Re}(\lambda)|-\epsilon+1/2} \cdot |b|^{-|\mathrm{Re}(\lambda)|-\epsilon+1/2} \cdot |c|^{|\mathrm{Re}(\lambda)|+\epsilon-1/2}. \end{aligned} \tag{9.5}$$

If π is of type (DS), let $\mathfrak{X}(\pi, \epsilon)$ be the set consisting of the following character of $\mathbf{T}(\mathbb{R})$:

$$\mathrm{diag}(a, b, ca^{-1}, cb^{-1}) \longmapsto |a|^{-\epsilon}|b|^{-\epsilon}|c|^{\epsilon}. \tag{9.6}$$

If π is of type (PS), let $\mathfrak{X}(\pi, \epsilon)$ be the set consisting of the following character of $\mathbf{T}(\mathbb{R})$:

$$\mathrm{diag}(a, b, ca^{-1}, cb^{-1}) \longmapsto |a|^{-|\mathrm{Re}(\lambda_1)|-|\mathrm{Re}(\lambda_2)|-2\epsilon}|c|^{(|\mathrm{Re}(\lambda_1)|+|\mathrm{Re}(\lambda_2)|)/2+\epsilon}. \tag{9.7}$$

We have the following uniform asymptotic estimate for Whittaker functions and an explicit construction of K-finite analytic families by (9.4).

Lemma 9.4. *(1) Let $\mathbf{W}_\lambda$ be a $\mathbf{K}^\square$-finite analytic family valued in $\mathcal{W}(\sigma_\lambda, \psi_{N^\square})$ and let $\varphi \in S(V^2(F))$. Then the map*

$$\lambda \longmapsto \mathcal{W}(\varphi, \mathbf{W}_\lambda) \tag{9.8}$$

defines a K-finite analytic family of Whittaker functions.

(2) Let $W \in \mathcal{W}(\pi, \psi_U)$. For any $\epsilon > 0$, there exist a non-negative function ϕ_ϵ on $(F^\times)^2$ and a constant $C_\epsilon > 0$ which satisfy the following conditions:

- $$|W(utk)| \leq \delta_{\mathbf{B}}(t)^{1/2}\phi_\epsilon(b^2c^{-1}, ab^{-1}) \sum_{\chi\in\mathfrak{X}(\pi,\epsilon)} \chi(t)$$

  for $t = \mathrm{diag}(a, b, ca^{-1}, cb^{-1}) \in \mathbf{T}(F)$, $u \in U(F)$ *and* $k \in K$.
- *When* π *is of type (IIa), we can extend* ϕ_ϵ *to a Schwartz function in* $\mathcal{S}(F^2)$ *of the form*

  $$\phi_\epsilon = C_\epsilon \cdot \mathbb{I}_{\varpi^{-n}\mathfrak{o}} \otimes \mathbb{I}_{\varpi^{-n}\mathfrak{o}}$$

  for some $n \in \mathbb{Z}$.
- *When* π *is of type (DS) or (PS), we have*

  $$\phi_\epsilon(a, b) \leq C_\epsilon \cdot e^{-\pi(|a|+|b|)/4}$$

  for $a, b \in \mathbb{R}^\times$.

(3) Let W_λ *be a* K*-finite analytic family of Whittaker functions. Then in (2), we can choose a function* $\phi_\epsilon = \phi_{\lambda,\epsilon}$ *for* W_λ *so that the constant* $C_\epsilon = C_{\lambda,\epsilon}$ *is bounded uniformly as* λ *varies in a compact set and the integer* n *is independent of* λ *in Case (IIa).*

Proof. To prove (2) for Case (DS), we may assume $W = \mathcal{W}(\varphi, \mathbf{W})$ for some $\mathbf{W} \in \mathcal{W}(\sigma, \psi_{N^\square})$ and $\varphi \in S(V^2(F))$. For Case (IIa) and Case (PS), we prove (1) and (2) simultaneously. Let $\mathbf{W}_\lambda$ be a $\mathbf{K}^\square$-finite analytic family valued in $\mathcal{W}(\sigma_\lambda, \psi_{N^\square})$ and let $\varphi \in S(V^2(F))$. We write $W_\lambda = \mathcal{W}(\varphi, \mathbf{W}_\lambda)$.

First we consider Case (IIa). Let $t = \mathrm{diag}(ab, a, b^{-1}, 1) \in \mathbf{T}(F)$ and $k \in K$. Choose $h_k \in H^\circ(\mathfrak{o})$ such that $\nu(h_k) = \nu(k)$. Then

$$W_\lambda(tk) = \int_{H_1^\circ(\mathfrak{o})} Z_\lambda^{(1)}(k_1; t, k)\, dk_1 + \int_{H_1^\circ(\mathfrak{o})} Z_\lambda^{(2)}(k_1; t, k)\, dk_1,$$

where

$$\begin{aligned} Z_\lambda^{(1)}(k_1; t, k) = |a| \int_F \int_{F^\times} \int_{F^\times} &\mathbf{W}_\lambda([\mathbf{d}(a)\mathbf{m}(y_1), \mathbf{u}(x)\mathbf{m}(y_2)]k_1h_k) \\ &\times \omega(tk, [\mathbf{d}(a)\mathbf{m}(y_1), \mathbf{u}(x)\mathbf{m}(y_2)]k_1h_k)\varphi(\mathbf{x}_0, \mathbf{y}_0)|y_1|^{-2}|y_2|^{-2}\, d^\times y_1\, d^\times y_2\, dx, \\ Z_\lambda^{(2)}(k_1; t, k) = |a\varpi^{-1}| \int_F \int_{F^\times} \int_{F^\times} &\mathbf{W}_\lambda([\mathbf{a}(\varpi), \mathbf{a}(\varpi)][\mathbf{d}(a)\mathbf{m}(y_1), \mathbf{u}(x)\mathbf{m}(y_2)]k_1h_k) \\ &\times \omega(tk, [\mathbf{a}(\varpi), \mathbf{a}(\varpi)][\mathbf{d}(a)\mathbf{m}(y_1), \mathbf{u}(x)\mathbf{m}(y_2)]k_1h_k)\varphi(\mathbf{x}_0, \mathbf{y}_0)|y_1|^{-2}|y_2|^{-2}\, d^\times y_1\, d^\times y_2\, dx. \end{aligned}$$

Let $k_1 \in H_1^\circ(\mathfrak{o})$. We have

$$\begin{aligned} Z_\lambda^{(1)}(k_1; t, k) = |a^3b^2| \int_{F^\times} \int_{F^\times} &\mathbf{W}_\lambda([\mathbf{d}(a)\mathbf{m}(y_1), \mathbf{m}(y_2)]k_1h_k)|y_1|^{-2}|y_2|^{-2} \\ &\times \int_F \psi(x)\omega(k, k_1h_k)\varphi\left(\begin{pmatrix} 0 & -aby_1^{-1}y_2^{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -ay_1^{-1}y_2 & -ay_1^{-1}y_2^{-1}x \\ 0 & y_1y_2^{-1} \end{pmatrix}\right) dx\, d^\times y_1\, d^\times y_2. \end{aligned}$$

Write

$$\omega(k, k_1h_k)\varphi(x, y) = \sum_i \varphi_0^{(i)}(x)\varphi_{11}^{(i)}(y_{11})\varphi_{12}^{(i)}(y_{12})\varphi_{21}^{(i)}(y_{21})\varphi_{22}^{(i)}(y_{22})$$

for some $\varphi_0^{(i)} \in \mathcal{S}(V(F))$, $\varphi_{11}^{(i)}, \varphi_{12}^{(i)}, \varphi_{21}^{(i)}, \varphi_{22}^{(i)} \in \mathcal{S}(F)$. Then

$$\begin{aligned} Z_\lambda^{(1)}(k_1; t, k) = |a^2b^2| \sum_i \varphi_{21}^{(i)}(0) \int_{F^\times} \int_{F^\times} &\mathbf{W}_\lambda([\mathbf{d}(a)\mathbf{m}(y_1), \mathbf{m}(y_2)]k_1h_k)|y_1|^{-1}|y_2|^{-1} \\ &\times \varphi_0^{(i)}\left(\begin{pmatrix} 0 & -aby_1^{-1}y_2^{-1} \\ 0 & 0 \end{pmatrix}\right) \varphi_{11}^{(i)}(-ay_1^{-1}y_2)\hat{\varphi}_{12}^{(i)}(-a^{-1}y_1y_2)\varphi_{22}^{(i)}(y_1y_2^{-1})\, d^\times y_1\, d^\times y_2, \end{aligned}$$

where $\hat{\varphi}_{12}^{(i)}$ is the Fourier transform of $\varphi_{12}^{(i)}$ with respect to ψ. Choose a constant $C > 0$ and a sufficiently large integer n independent of k, k_1, and λ such that $\sum_i |\varphi_{21}^{(i)}(0)| \leq C$ and

$$|\varphi_0^{(i)}| \leq C \cdot \mathbb{I}_{V(\varpi^{-n}\mathfrak{o})}, \quad |\varphi_{11}^{(i)}| \leq C \cdot \mathbb{I}_{\varpi^{-n}\mathfrak{o}}, \quad |\hat{\varphi}_{12}^{(i)}| \leq C \cdot \mathbb{I}_{\varpi^{-n}\mathfrak{o}}, \quad |\varphi_{22}^{(i)}| \leq C \cdot \mathbb{I}_{\varpi^{-n}\mathfrak{o}}$$

for all i. Then

$$|Z^{(1)}_\lambda(k_1;t,k)| \le C^5 \cdot \delta_{\mathbf{B}}(t)^{1/2} \int_{F^\times}\int_{F^\times} |\mathbf{W}_\lambda([\mathbf{d}(a)\mathbf{m}(y_1),\mathbf{m}(y_2)]k_1h_k)||a|^{1/2}|y_1|^{-1}|y_2|^{-1} \times \mathbb{I}_{\varpi^{-n}\mathfrak{o}}(aby_1^{-1}y_2^{-1})\mathbb{I}_{\varpi^{-n}\mathfrak{o}}(ay_1^{-1}y_2)\mathbb{I}_{\varpi^{-n}\mathfrak{o}}(a^{-1}y_1y_2)\mathbb{I}_{\varpi^{-n}\mathfrak{o}}(y_1y_2^{-1})\, d^\times y_1\, d^\times y_2.$$

Let $\epsilon > 0$. There exists $\Phi_{\lambda,\epsilon} \in \mathcal{S}(F^2)$ such that

$$|\mathbf{W}_\lambda([\mathbf{a}(t_1),\mathbf{a}(t_2)]k')| \le |t_1|^{-|\mathrm{Re}(\lambda)|+1/2-\epsilon}|t_2|^{1/2}\Phi_{\lambda,\epsilon}(t_1,t_2)$$

for $t_1, t_2 \in F^\times$ and $k' \in H^\circ(\mathfrak{o})$. Moreover, we can find such $\Phi_{\lambda,\epsilon}$ which is bounded uniformly as λ varies in a compact set and whose support is contained in $\varpi^{-2n}\mathfrak{o} \times \varpi^{-2n}\mathfrak{o}$ for some integer n independent of λ. Hence it suffices to consider the integral

$$\int_{F^\times}\int_{F^\times} \chi_{\lambda,\epsilon}(a^{-1}y_1^2)|y_2|\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(a^{-1}y_1^2)\mathbb{I}_{\varpi^{-n}\mathfrak{o}}(y_2) \times \mathbb{I}_{\varpi^{-n}\mathfrak{o}}(aby_1^{-1}y_2^{-1})\mathbb{I}_{\varpi^{-n}\mathfrak{o}}(ay_1^{-1}y_2)\mathbb{I}_{\varpi^{-n}\mathfrak{o}}(a^{-1}y_1y_2)\mathbb{I}_{\varpi^{-n}\mathfrak{o}}(y_1y_2^{-1})\, d^\times y_1\, d^\times y_2, \tag{9.9}$$

where $\chi_{\lambda,\epsilon} = |\ |^{-|\mathrm{Re}(\lambda)|-\epsilon}$. The integral is majorized by

$$\left[C^{(1)}_{\lambda,\epsilon} + C^{(2)}_{\lambda,\epsilon}\cdot|ab|^{1/2} + C^{(3)}_{\lambda,\epsilon}\cdot\chi_{\lambda,\epsilon}(ab) + C^{(4)}_{\lambda,\epsilon}\cdot\chi_{\lambda,\epsilon}(ab)|ab|^{1/2}\right]\cdot\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(a)\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(b)$$

for some constants $C^{(1)}_{\lambda,\epsilon}, C^{(2)}_{\lambda,\epsilon}, C^{(3)}_{\lambda,\epsilon}, C^{(4)}_{\lambda,\epsilon}$ which are bounded uniformly as λ varies in a compact set. Indeed, it is clear that (9.9) is majorized by

$$\int_{F^\times} \chi_{\lambda,\epsilon}(a^{-1}y_1^2)\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(a^{-1}y_1^2)\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(a^2by_1^{-2})\, d^\times y_1 \cdot \int_{F^\times} |y_2|\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(y_2^2)\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(aby_2^{-2})\, d^\times y_2 \times \mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(a)\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(b).$$

Assume that both $\log_q|a|$ an $\log_q|b|$ are even integers. The other cases can be treated in a similar way. Then the above integral is equal to

$$(\chi_{\lambda,\epsilon}(\varpi^2)-1)^{-1}(\chi_{\lambda,\epsilon}(ab\varpi^{2n+2}) - \chi_{\lambda,\epsilon}(\varpi^{-2n}))\cdot(q^{-1}-1)^{-1}(q^{-n-1}|ab|^{1/2} - q^n)\cdot\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(a)\mathbb{I}_{\varpi^{-2n}\mathfrak{o}}(b).$$

Therefore we obtain a uniform estimate for $Z^{(1)}_\lambda(k_1;t,k)$. We have a similar estimate for $Z^{(2)}_\lambda(k_1;t,k)$. Moreover, it follows from the above estimate that the map $\lambda \mapsto W_\lambda(g)$ is analytic for each $g \in G(F)$. The continuity of the map $(\lambda, g) \mapsto W_\lambda(g)$ can be proved in a similar way. Therefore the map $\lambda \mapsto W_\lambda$ defines a K-finite analytic family. This completes the proof of (1) and (2) for Case (IIa).

Next we consider Case (PS). Let $t = \mathrm{diag}(ab, a, b^{-1}, 1) \in \mathbf{T}(\mathbb{R})$ and $k \in K$. Choose $h_k \in \mathbf{K}^\square$ such that $\nu(h_k) = \nu(k)$. Then

$$W_\lambda(tk) = \int_{\mathbf{K}^\square \cap H_1^\circ(\mathbb{R})} Z_\lambda(k_1;t,k)\, dk_1,$$

where

$$\begin{aligned}Z_\lambda(k_1;t,k) &= |a|\int_{\mathbb{R}}\int_0^\infty\int_0^\infty \mathbf{W}_\lambda([\mathbf{d}(a)\mathbf{m}(y_1),\mathbf{u}(x)\mathbf{m}(y_2)]k_1h_k) \\ &\quad \times \omega(tk,[\mathbf{d}(a)\mathbf{m}(y_1),\mathbf{u}(x)\mathbf{m}(y_2)]k_1h_k)\varphi(\mathbf{x}_0,\mathbf{y}_0)y_1^{-2}y_2^{-2}\, d^\times y_1\, d^\times y_2\, dx \\ &= \delta_{\mathbf{B}}(t)^{1/2}\int_0^\infty\int_0^\infty \mathbf{W}_\lambda([\mathbf{d}(a)\mathbf{m}(y_1),\mathbf{m}(y_2)]k_1h_k)|a|^{1/2}y_1^{-1}y_2^{-1} \\ &\quad \times \int_{\mathbb{R}}\psi(-a^{-1}y_1y_2x)\omega(k,k_1h_k)\varphi\left(\begin{pmatrix}0 & -aby_1^{-1}y_2^{-1}\\ 0 & 0\end{pmatrix},\begin{pmatrix}-ay_1^{-1}y_2 & -x\\ 0 & y_1y_2^{-1}\end{pmatrix}\right)dx\, d^\times y_1\, d^\times y_2.\end{aligned}$$

The inner integral over $\mathbb{R}$ is equal to the product of $e^{-\pi(y_1^2y_2^{-2}+a^2y_1^{-2}y_2^2+a^2b^2y_1^{-2}y_2^{-2}+a^{-2}y_1^2y_2^2)}$ and a polynomial in four variables $y_1y_2^{-1}$, $ay_1^{-1}y_2$, $aby_1^{-1}y_2^{-1}$, and $a^{-1}y_1y_2$. Note that this polynomial is independent of λ. We conclude that

$$|Z_\lambda(k_1;t,k)| \ll_{k,k_1} \delta_{\mathbf{B}}(t)^{1/2}\int_0^\infty\int_0^\infty |\mathbf{W}_\lambda([\mathbf{d}(a)\mathbf{m}(y_1),\mathbf{m}(y_2)]k_1h_k)||a|^{1/2}y_1^{-1}y_2^{-1} \times e^{-\pi(y_1^2y_2^{-2}+a^2y_1^{-2}y_2^2+a^2b^2y_1^{-2}y_2^{-2}+a^{-2}y_1^2y_2^2)/2}\, d^\times y_1\, d^\times y_2.$$

For $\epsilon_1, \epsilon_2 > 0$, we have

$$|\mathbf{W}_\lambda([\mathbf{a}(t_1), \mathbf{a}(t_2)]k')| \ll_{\lambda_1,\lambda_2,\epsilon_1,\epsilon_2} |t_1|^{-\mu_1+1/2-\epsilon_1}|t_2|^{-\mu_2+1/2-\epsilon_2}e^{-\pi(|t_1|+|t_2|)}$$

for $t_1, t_2 \in \mathbb{R}^\times$ and $k' \in \mathbf{K}^\square$. Here

$$\mu_1 = \frac{|\mathrm{Re}(\lambda_1)| + |\mathrm{Re}(\lambda_2)|}{2}, \quad \mu_2 = \frac{||\mathrm{Re}(\lambda_1)| - |\mathrm{Re}(\lambda_2)||}{2}.$$

Assume $\mu_1 \geq \mu_2$. The case $\mu_2 \leq \mu_2$ can be proved in a similar way and we omit it. Let $\epsilon_1 = 2\epsilon_2 = \epsilon > 0$. Then

$$\begin{aligned}
&\int_0^\infty \int_0^\infty |\mathbf{W}_\lambda([\mathbf{d}(a)\mathbf{m}(y_1), \mathbf{m}(y_2)]k_1h_k)||a|^{1/2}y_1^{-1}y_2^{-1}e^{-\pi(y_1^2y_2^{-2}+a^2y_1^{-2}y_2^2+a^2b^2y_1^{-2}y_2^{-2}+a^{-2}y_1^2y_2^2)/2}\,d^\times y_1\,d^\times y_2\\
&\ll_{\lambda_1,\lambda_2,\epsilon} |a|^{\mu_1+\epsilon}\int_0^\infty\int_0^\infty y_1^{-2\mu_1-2\epsilon}y_2^{-2\mu_2-\epsilon}e^{-\pi y_1^2(a^{-2}y_2^2+y_2^{-2})/2-\pi y_1^{-2}(a^2y_2^2+a^2b^2y_2^{-2})/2}\,d^\times y_1\,d^\times y_2\\
&\ll_{\lambda_1,\lambda_2,\epsilon} |a|^{\mu_1+\epsilon}\int_0^\infty y^{-2\mu_2-\epsilon}\left(a^{-2}y^2+y^{-2}\right)^{(\mu_1+\epsilon)/2}\left(a^2y^2+a^2b^2y^{-2}\right)^{-(\mu_1+\epsilon)/2}\\
&\qquad\qquad \times K_{\mu_1+\epsilon}\left(\pi(a^{-2}y^2+y^{-2})^{1/2}(a^2y^2+a^2b^2y^{-2})^{1/2}\right)\,d^\times y\\
&\ll_{\lambda_1,\lambda_2,\epsilon} |a|^{\mu_1+\epsilon}\int_0^\infty y^{-2\mu_2-\epsilon}\left(a^2y^2+a^2b^2y^{-2}\right)^{-(\mu_1+\epsilon)}e^{-\pi(a^{-2}y^2+y^{-2})^{1/2}(a^2y^2+a^2b^2y^{-2})^{1/2}/2}\,d^\times y.
\end{aligned}$$

Here, in the last two inequalities, we use the integral representation (2.1) of $K_{\mu_1+\epsilon}$ and the estimate

$$|K_{\alpha_1}(y)| \ll_{\alpha_1,\alpha_2} y^{-\alpha_2}e^{-y/2} \tag{9.10}$$

for $0 < \alpha_1 \leq \alpha_2$ and $y > 0$. By the inequality

$$\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$$

for $x, y \geq 0$, we have

$$\begin{aligned}
&\int_0^\infty y^{-2\mu_2-\epsilon}\left(a^2y^2+a^2b^2y^{-2}\right)^{-(\mu_1+\epsilon)}e^{-\pi(a^{-2}y^2+y^{-2})^{1/2}(a^2y^2+a^2b^2y^{-2})^{1/2}/2}\,d^\times y\\
&\ll_{\lambda_1,\lambda_2,\epsilon} |a|^{-2\mu_1-2\epsilon}|b|^{-2\mu_1-2\epsilon}\int_0^\infty y^{\mu_1-\mu_2+\epsilon/2}e^{-\pi(|a|+|b|+y+|ab|y^{-1})/4}\,d^\times y.
\end{aligned}$$

By the integral representation (2.1) of $K_{\mu_1-\mu_2+\epsilon/2}$, the estimate (9.10), and the assumption that $\mu_1 \geq \mu_2$, we have

$$\begin{aligned}
&\int_0^\infty y^{\mu_1-\mu_2+\epsilon/2}e^{-\pi(|a|+|b|+y+|ab|y^{-1})/4}\,d^\times y\\
&\ll_{\lambda_1,\lambda_2,\epsilon} e^{-\pi(|a|+|b|)/4}|ab|^{\mu_1/2-\mu_2/2+\epsilon/4}K_{\mu_1-\mu_2+\epsilon/2}(2^{-1}\pi|ab|^{1/2})\\
&\ll_{\lambda_1,\lambda_2,\epsilon} e^{-\pi(|a|+|b|)/4}.
\end{aligned}$$

Moreover, we can choose constants in the above inequalities so that they are bounded uniformly as λ varies in a compact set. Therefore we obtain a uniform estimate for $Z_\lambda(k_1; t, k)$. A similar estimate shows that the map $\lambda \mapsto W_\lambda$ is a K-finite analytic family. This completes the proof of (1) and (2) for Case (PS).

Finally we assume π is of type (DS). There exists a polynomial $P \in \mathbb{C}[X, Y]$ divisible by XY such that

$$|W([\mathbf{a}(t_1), \mathbf{a}(t_2)]k')| \leq P(|t_1|, |t_2|)e^{-2\pi(|t_1|+|t_2|)}$$

for $t_1, t_2 \in \mathbb{R}^\times$ and $k' \in \mathbf{K}^\square$. Similarly as in Case (PS), we only need to consider the integral

$$\int_0^\infty\int_0^\infty P(|a^{-1}y_1^2|, |y_2|^2)|a|^{1/2}y_1^{-1}y_2^{-1}e^{-\pi(y_1^2y_2^{-2}+a^2y_1^{-2}y_2^2+a^2b^2y_1^{-2}y_2^{-2}+a^{-2}y_1^2y_2^2)/2}\,d^\times y_1\,d^\times y_2.$$

We may assume $P(X,Y) = X^m Y^n$ for some $m, n \in \mathbb{Z}_{\geq 1}$. By an estimate similar to the one in Case (PS), we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty P(|a^{-1}y_1^2|, |y_2|^2)|a|^{1/2} y_1^{-1} y_2^{-1} e^{-\pi(y_1^2 y_2^{-2} + a^2 y_1^{-2} y_2^2 + a^2 b^2 y_1^{-2} y_2^{-2} + a^{-2} y_1^2 y_2^2)/2}\, d^\times y_1\, d^\times y_2 \\
&\ll_{m,n} |a|^{-m+1/2} \int_0^\infty y^{2n-1} \left(a^2 y^2 + a^2 b^2 y^{-2}\right)^{(2m-1)/4} \left(a^{-2} y^2 + y^{-2}\right)^{-(2m-1)/4} \\
&\qquad\qquad \times K_{m-1/2}\left(\pi (a^{-2} y^2 + y^{-2})^{1/2} (a^2 y^2 + a^2 b^2 y^{-2})^{1/2}\right) d^\times y \\
&\ll_{m,n} |a|^{-m+1/2} \int_0^\infty y^{2n-1} \left(a^{-2} y^2 + y^{-2}\right)^{-(2m-1)/2} e^{-\pi (a^{-2} y^2 + y^{-2})^{1/2} (a^2 y^2 + a^2 b^2 y^{-2})^{1/2}/2}\, d^\times y \\
&\ll_{m,n} \int_0^\infty y^{n-1/2} e^{-\pi(|a| + |b| + y + |ab| y^{-1})/4}\, d^\times y \\
&\ll_{m,n} e^{-\pi(|a|+|b|)/4} |ab|^{(2n-1)/4} K_{n-1/2}(2^{-1}\pi |ab|^{1/2}) \\
&\ll_{m,n,\epsilon} |ab|^{-\epsilon} e^{-\pi(|a|+|b|)/4}.
\end{aligned}$$

Here the last inequality follows from (9.10) with $\alpha_1 = n - 1/2$ and $\alpha_2 = n - 1/2 + 2\epsilon$. This completes the proof of (2) for Case (DS).

It remains to prove (3). Note that for any fixed $\lambda_0 \in \Omega$, any K-finite analytic family of Whittaker functions can be written as a linear combination, with analytic functions of λ as coefficients, of K-finite analytic families of the form (9.8) in a neighborhood of λ_0. Since we only consider the convergence for λ varying in a compact set, it suffices to consider K-analytic families of the form (9.8). It follows from the above estimate for Case (IIa) and Case (PS) that (3) holds for K-analytic families of the form (9.8). This completes the proof. □

Lemma 9.5. *Let $\mathcal{F} \in \mathcal{I}(s)$ be a holomorphic section.*

(1) Let $W \in \mathcal{W}(\pi, \psi_U)$. The integral $\mathcal{Z}(s, W, \mathcal{F})$ is absolutely convergent for

$$\begin{cases} \mathrm{Re}(s) > -1 + 4|\mathrm{Re}(\lambda)| & \text{if } \pi \text{ is of type (IIa)}, \\ \mathrm{Re}(s) > -1 & \text{if } \pi \text{ is of type (DS)}, \\ \mathrm{Re}(s) > -1 + 2(|\mathrm{Re}(\lambda_1)| + |\mathrm{Re}(\lambda_2)|) & \text{if } \pi \text{ is of type (PS)}. \end{cases}$$

In particular, the integral is absolutely convergent for $\mathrm{Re}(s) \geq 1$ if π is of type (IIa) or (PS) with parameter in $\mathcal{D}$.

(2) Let W_λ be a K-finite analytic family of Whittaker functions. The integral $\mathcal{Z}(s, \mathcal{W}(\lambda), \mathcal{F})$ is uniformly convergent as λ varies in a compact set.

Proof. We prove (1) and (2) simultaneously. Let

$$(K \times K)^\circ = \{(k_1, k_2) \in K \times K \mid \nu(k_1) = \nu(k_2)\}.$$

We define $(\mathbf{B} \times \mathbf{B})^\circ$ and $(\mathbf{T} \times \mathbf{T})^\circ$ in a similar way. Let

$$U' = \left\{ \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{G}_a \right\}$$

be a unipotent subgroup of G. For $W \in \mathcal{W}(\pi, \psi_U)$, let

$$W'(g) = \begin{cases} W(\mathrm{diag}(-1,1,1,-1)g) & \text{if } \pi \text{ is of type (IIa) or (PS)}, \\ \overline{W(g)} & \text{if } \pi \text{ is of type (DS)}. \end{cases}$$

We have

$$\begin{aligned}\mathcal{Z}(s,W,\mathcal{F}) &= \int_{Z_{\mathrm{GSp}_8}(F)\tilde{U}(F)\backslash \mathbf{G}(F)} \mathcal{F}(\eta g,s)(W\otimes W')(g)\,dg \\ &= \int_{(K\times K)^\circ}\int_{Z_{\mathrm{GSp}_8}(F)\backslash(\mathbf{T}\times\mathbf{T})^\circ(F)} \delta_{(\mathbf{B}\times\mathbf{B})^\circ}(t)^{-1}(W\otimes W')(tk) \\ &\qquad\qquad \times \int_{U'(F)\backslash U(F)} \mathcal{F}(\eta(u,1)tk)\psi_U(u)\,du\,dt\,dk.\end{aligned}$$

Let $s\in\mathbb{R}$ and $\epsilon>0$. Since

$$\left\{\left(\mathrm{diag}(ab,a,b^{-1},1),\mathrm{diag}(cd,c,c^{-1}d^{-1}a,c^{-1}a)\right)\ \mid a,b,c,d\in F^\times\right\}$$

is a set of representatives for $Z_{\mathrm{GSp}_8}(F)\backslash(\mathbf{T}\times\mathbf{T})^\circ(F)$, by Lemma 9.4, it suffices to consider the integrals

$$\begin{aligned}&\int_{(F^\times)^4}\delta_{(\mathbf{B}\times\mathbf{B})^\circ}((t_1,t_2))^{-1/2}\phi_\epsilon(a,b)\phi_\epsilon(c^2a^{-1},d)\chi_1(t_1)\chi_2(t_2)\\ &\qquad\qquad \times\int_{U'(F)\backslash U(F)}\mathcal{F}(\eta(u,1)(t_1,t_2))\,du\,d(a,b,c,d)\end{aligned}$$

for $\chi_1,\chi_2\in\mathfrak{X}(\pi,\epsilon)$. Here ϕ_ϵ is a function on $(F^\times)^2$ satisfying the conditions in Lemma 9.4-(2), $d(a,b,c,d)$ is a Haar measure on $(F^\times)^4$, and

$$t_1=\mathrm{diag}(ab,a,b^{-1},1),\quad t_2=\mathrm{diag}(cd,c,c^{-1}d^{-1}a,c^{-1}a).$$

We may assume $\mathcal{F}$ is $\mathrm{GSp}_8(\mathfrak{o})$-invariant (resp. $\mathrm{GSp}_8(\mathbb{R})\cap\mathrm{O}(8)$-invariant) when F is non-archimedean (resp. $F=\mathbb{R}$) and $\mathcal{F}(1,s)=1$. By [Jia96, p. 173 and 177, (198) and (214)], we have

$$\begin{aligned}&\int_{U'(F)\backslash U(F)}\mathcal{F}(\eta(u,1)(t_1,t_2))\,du\\ &=\delta_{(\mathbf{B}\times\mathbf{B})^\circ}((t_1,t_2))^{1/2}|a^{3/2}bc^{-1}|^{s+1}\frac{\zeta(s+1)^2}{\zeta(s+2)\zeta(s+3)}f^o(u,v,s),\end{aligned}$$

where

$$u=abc^{-1}d^{-1},\qquad v=ac^{-1},$$

and

$$f^o(u,v,s)=\begin{cases}\max\{1,|u|\}^{-s-1}\max\{1,|v|\}^{-2s-2} & \text{if } F \text{ is non-archimedean},\\ (1+u^2)^{-(s+1)/2}(1+v^2)^{-s-1} & \text{if } F=\mathbb{R}.\end{cases}$$

Fix $\chi_1,\chi_2\in\mathfrak{X}(\pi,\epsilon)$. Let $k_1,k_2,k_3,k_1',k_2',k_3'\in\mathbb{R}$ be such that

$$\chi_1(t)=|a|^{k_1}|b|^{k_2}|c|^{k_3},\quad \chi_2(t)=|a|^{k_1'}|b|^{k_2'}|c|^{k_3'}$$

for $t=\mathrm{diga}(a,b,ca^{-1},cb^{-1})$. Then

$$\chi_1(t_1)\chi_2(t_2)=|a|^{k_a}|b|^{k_b}|c|^{k_c}|d|^{k_d},$$

where

$$k_a=k_1+k_2+k_3+k_3',\qquad k_b=k_1,\qquad k_c=k_1'+k_2',\qquad k_d=k_1'.$$

Hence it suffices to consider the integral

$$\int_{(F^\times)^4}\phi_{\chi_1}(a,b)\phi_{\chi_2}(c^2a^{-1},d)\cdot|a|^{3s/2+k_a+3/2}|b|^{s+k_b+1}|c|^{-s+k_c-1}|d|^{k_d}f^o(u,v,s)\,d(a,b,c,d).$$

We change variables $(a,b,c,d)\mapsto(a,b,u,v)$ to get

$$\int_{(F^\times)^4}\phi_{\chi_1}(a,b)\phi_{\chi_2}(av^{-2},bu^{-1}v)\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{l_u}|v|^{s+l_v+1}f^o(u,v,s)\,d(a,b,u,v), \tag{9.11}$$

where

$$\begin{aligned} l_a &= k_a + k_c = k_1 + k_2 + k_3 + k_1' + k_2' + k_3', \\ l_b &= k_b + k_d = k_1 + k_1', \\ l_u &= -k_d = -k_1', \\ l_v &= -k_c + k_d = -k_2'. \end{aligned}$$

First we consider Case (IIa). By Lemma 9.4-(3), we can choose a Schwartz function $\phi_\epsilon = \phi_{\lambda,\epsilon}$ for W_λ of the form

$$\phi_{\lambda,\epsilon}(a,b) = C_{\lambda,\epsilon} \cdot \mathbb{I}_{\varpi^{-n}\mathfrak{o}}(a)\mathbb{I}_{\varpi^{-n}\mathfrak{o}}(b).$$

so that the constant $C_{\lambda,\epsilon} > 0$ is bounded uniformly as λ varies in a compact set and the integer n is independent of λ. Put $r = q^n > 1$ and $\phi = \mathbb{I}_{\varpi^{-n}\mathfrak{o}}$. We write the integral (9.11) as

$$C_{\lambda,\epsilon}^2 \cdot (I_1 + I_2 + I_3 + I_4),$$

where

$$\begin{aligned} I_1 &= \int_{F^\times}\int_{F^\times}\int_{|u|\le 1}\int_{|v|\le 1} \phi(a)\phi(b)\phi(av^{-2})\phi(bu^{-1}v)\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{l_u}|v|^{s+l_v+1}\,d(a,b,u,v) \\ &= \int_{|a|\le r}\int_{|b|\le r}\int_{|u|\le 1}\int_{|v|\le 1} \phi(av^{-2})\phi(bu^{-1}v)\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{l_u}|v|^{s+l_v+1}\,d(a,b,u,v), \\ I_2 &= \int_{F^\times}\int_{F^\times}\int_{|u|\le 1}\int_{|v|> 1} \phi(a)\phi(b)\phi(av^{-2})\phi(bu^{-1}v)\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{l_u}|v|^{-s+l_v-1}\,d(a,b,u,v) \\ &= \int_{|a|\le r}\int_{|b|\le r}\int_{|u|\le 1}\int_{|v|< 1} \phi(av^{2})\phi(bu^{-1}v^{-1})\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{l_u}|v|^{s-l_v+1}\,d(a,b,u,v), \\ I_3 &= \int_{F^\times}\int_{F^\times}\int_{|u|> 1}\int_{|v|\le 1} \phi(a)\phi(b)\phi(av^{-2})\phi(bu^{-1}v)\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{-s+l_u-1}|v|^{s+l_v+1}\,d(a,b,u,v) \\ &= \int_{|a|\le r}\int_{|b|\le r}\int_{|u|< 1}\int_{|v|\le 1} \phi(av^{-2})\phi(buv)\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{s-l_u+1}|v|^{s+l_v+1}\,d(a,b,u,v), \\ I_4 &= \int_{F^\times}\int_{F^\times}\int_{|u|> 1}\int_{|v|> 1} \phi(a)\phi(b)\phi(av^{-2})\phi(bu^{-1}v)\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{-s+l_u-1}|v|^{-s+l_v-1}\,d(a,b,u,v) \\ &= \int_{|a|\le r}\int_{|b|\le r}\int_{|u|< 1}\int_{|v|< 1} \phi(av^{2})\phi(buv^{-1})\cdot|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{s-l_u+1}|v|^{s-l_v+1}\,d(a,b,u,v). \end{aligned}$$

We assume $l_u \neq 0$. The case $l_u = 0$ can be proved in a similar way and we omit it. For $|b| \le r$ and $|v| \le 1$, we have

$$\int_{|u|\le 1}\phi(bu^{-1}v)|u|^{l_u}\,du = \int_{r^{-1}|bv|\le|u|\le 1}|u|^{l_u}\,du = \frac{1-(q^{-1}r^{-1}|bv|)^{l_u}}{1-q^{-l_u}}.$$

Then

$$\begin{aligned} |1-q^{-l_u}|\cdot I_1 \le& \int_{|a|\le r}\int_{|b|\le r}\int_{|v|\le 1}\phi(av^{-2})|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|v|^{s+l_v+1}\,d(a,b,v) \\ &+ q^{-l_u}r^{-l_u}\int_{|a|\le r}\int_{|b|\le r}\int_{|v|\le 1}\phi(av^{-2})|a|^{s/2+l_a+1/2}|b|^{s+l_b+l_u+1}|v|^{s+l_u+l_v+1}\,d(a,b,v). \end{aligned}$$

Similarly, we have

$$\int_{|u|\le 1}\phi(bu^{-1}v^{-1})|u|^{l_u}\,du = \frac{1-(q^{-1}r^{-1}|bv^{-1}|)^{l_u}}{1-q^{-l_u}}$$

if $r^{-1}|bv^{-1}| \le 1$ and

$$\int_{|u|\le 1}\phi(bu^{-1}v^{-1})|u|^{l_u}\,du = 0$$

otherwise. Hence we have

$$
\begin{aligned}
|1-q^{-l_u}|\cdot I_2 \le & \int_{|a|\le r}\int_{|b|\le r}\int_{|v|<1} |a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|v|^{s-l_v+1}\,d(a,b,v)\\
&+ q^{-l_u}r^{-l_u}\int_{|a|\le r}\int_{|b|\le r}\int_{|v|<1} |a|^{s/2+l_a+1/2}|b|^{s+l_b+l_u+1}|v|^{s-l_u-l_v+1}\,d(a,b,v).
\end{aligned}
$$

We also have

$$
\begin{aligned}
I_3 &\le \int_{|a|\le r}\int_{|b|\le r}\int_{|u|<1}\int_{|v|\le 1} \phi(av^{-2})|a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{s-l_u+1}|v|^{s+l_v+1}\,d(a,b,u,v),\\
I_4 &\le \int_{|a|\le r}\int_{|b|\le r}\int_{|u|<1}\int_{|v|<1} |a|^{s/2+l_a+1/2}|b|^{s+l_b+1}|u|^{s-l_u+1}|v|^{s-l_v+1}\,d(a,b,u,v).
\end{aligned}
$$

Note that the integrals

$$
\int_{|a|\le r}\int_{|v|\le 1}\phi(av^{-2})|a|^{s/2+l_a+1/2}|v|^{s+l_v+1}\,d(a,v),\qquad \int_{|a|\le r}\int_{|v|\le 1}\phi(av^{-2})|a|^{s/2+l_a+1/2}|v|^{s+l_u+l_v+1}\,d(a,v)
$$

are absolutely convergent for

$$
s>\max\{-2l_a-1,-l_a-l_v/2-1\},\quad s>\max\{-2l_a-1,-l_a-l_u/2-l_v/2-1\},
$$

respectively. We conclude that the integrals I_1, I_2, I_3, and I_4 are absolutely convergent for

$$
s>\max\left\{-2l_a-1,-l_b-1,l_u-1,l_v-1,-l_b-l_u-1,l_u+l_v-1,-l_a-l_v/2-1,-l_a-l_u/2-l_v/2-1\right\}.
$$

From (9.5), one can verify that the above inequality holds if $s>4|\mathrm{Re}(\lambda)|+4\epsilon-1$. Moreover, the above integrals are uniformly convergent as λ varies in a compact set. This completes the proof for Case (IIa).

Next we consider case (PS). Put

$$
\mu=\frac{|\mathrm{Re}(\lambda_1)|+|\mathrm{Re}(\lambda_2)|}{2}.
$$

By (9.7),

$$
l_a=-2\mu-2\epsilon,\quad l_b=-4\mu-4\epsilon,\quad l_u=2\mu+2\epsilon,\quad l_v=0.
$$

Assume $s>4\mu+4\epsilon-1$. By Lemma 9.4-(3), we can choose a function $\phi_\epsilon=\phi_{\lambda,\epsilon}$ for W_λ satisfying

$$
\phi_{\lambda,\epsilon}(a,b)\le C_{\lambda,\epsilon}\cdot e^{-\pi(|a|+|b|)/4}
$$

for $a,b\in\mathbb{R}^\times$ so that the constant $C_{\lambda,\epsilon}>0$ is bounded uniformly as λ varies in a compact set. Then the integral (9.11) is bounded by

$$
C_{\lambda,\epsilon}^2\int_{(\mathbb{R}^\times)^4}|a|^{s/2+1/2-2(\mu+\epsilon)}|b|^{s+1-4(\mu+\epsilon)}e^{-\pi(|a|+|b|)/4}\frac{|u|^{2(\mu+\epsilon)}}{(1+u^2)^{(s+1)/2}}\frac{|v|^{s+1}}{(1+v^2)^{s+1}}\,d(a,b,u,v),
$$

which is absolutely convergent. Moreover, it is clear that the above integral is uniformly convergent as λ varies in a compact set. This completes the proof for Case (PS).

Finally we assume π is of type (DS). By (9.6),

$$
l_a=-2\epsilon,\quad l_b=-2\epsilon,\quad l_u=\epsilon,\quad l_v=\epsilon.
$$

Assume $s>4\epsilon-1$. By Lemma 9.4-(2), there exists a constant $C_\epsilon>0$ such that

$$
\phi_\epsilon(a,b)\le C_\epsilon\cdot e^{-\pi(|a|+|b|)/4}
$$

for $a,b\in\mathbb{R}^\times$. Then the integral (9.11) is bounded by

$$
C_\epsilon^2\int_{(\mathbb{R}^\times)^4}|a|^{s/2+1/2-2\epsilon}|b|^{s+1-2\epsilon}e^{-\pi(|a|+|b|)/4}\frac{|u|^{\epsilon}}{(1+u^2)^{(s+1)/2}}\frac{|v|^{s+1+\epsilon}}{(1+v^2)^{s+1}}\,d(a,b,u,v),
$$

which is absolutely convergent. This completes the proof. □

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