CORRIGENDUM TO "ON THE $1/H$-FLOW BY p-LAPLACE APPROXIMATION: NEW ESTIMATES VIA FAKE DISTANCES UNDER RICCI LOWER BOUNDS [*AMER. J. MATH.* 144 (2022), NO. 3, 779–849]"

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Abstract. We correct a mistake in our proof of Lemma 2.17. Although we have to strengthen the assumptions therein and, accordingly, in Theorem 2.22, all of the results on the existence and properties of the IMCF are not affected. Minor changes, with no influence elsewhere in the paper, regard Lemma 3.3, Proposition 4.3 and Lemma 5.3.

Lemma 2.17 needs to be replaced by the following statement:

LEMMA 0.1 (Key lemma). *Let M^m be complete. Let $p \in (1,\infty)$, and let $0 \le H \in C(\mathbb{R}_0^+)$ be non-increasing. Consider a model M_h with radial curvature $-H(r)$, and assume that Δ_p is non-parabolic on M_h. Let u be a positive solution to $\Delta_p u = 0$ in an open set $\Omega \subset M$, possibly the entire M, and define ϱ according to*

$$u(x) = \mathscr{G}^h\big(\varrho(x)\big) = \int_{\varrho(x)}^{\infty} v_h(s)^{-\frac{1}{p-1}} \mathrm{d}s.$$

When $p > m$, also suppose that $u < \mathscr{G}^h(0)$ on Ω. Assume that

$$\text{(0.1)} \qquad \text{(i) } \inf_M \mathrm{Ric} > -\infty, \quad \text{(ii) } \mathrm{Ric} \ge -(m-1)H(\varrho) \quad \text{on } \Omega$$

and that either

(a) $m = 2 \le p$, *or*

(b) (a) *fails and one of the following conditions is satisfied for some sequence $R_j \to \infty$:*

$$\log \|u\|_{L^\infty(\Omega \cap B_{R_j})} = o(R_j^2) \quad \text{if } p < m,$$
$$\|u\|_{L^\infty(\Omega \cap B_{R_j})} = o(R_j^2) \quad \text{if } p = m,$$
$$\|u\|_{L^\infty(\Omega)} < \mathscr{G}^h(0) \quad \text{if } p > m.$$

Then,

$$\text{(0.2)} \qquad \sup_\Omega |\nabla \varrho| \le \max\left\{1, \limsup_{x \to \partial\Omega} |\nabla \varrho(x)|\right\},$$

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where we set

$$\limsup_{x\to\partial\Omega}|\nabla\varrho(x)| \doteq \inf\Big\{\sup_{\Omega\setminus\overline{V}}|\nabla\varrho| : V \textit{ open whose closure in } M \textit{ satisfies } \overline{V}\subset\Omega\Big\}.$$

In particular, if $\partial\Omega=\emptyset$ *then* $|\nabla\varrho|\le 1$.

Remark 0.2. The new statement and proof of Lemma 2.17 should also replace Lemma 4.13 in the book [1], where no further related modification is needed.

In [2], inequality (0.2) was claimed to hold without the restrictions in (a), (b). Although we needed them to overcome the problem explained below, still it seems reasonable to us that Lemma 2.17 should hold in its original form. We also make explicit the completeness of M and (0.1) (i), which were only implicitly assumed. These two requirements should also appear in [2, Rem. 2.18 (1)].

The proof of Lemma 2.17 was by contradiction: assuming that (0.2) fails and that $H_*\doteq\inf H>0$, we obtained a growth estimate for the quantity

$$I(R)\doteq\int_{B_R}\lambda h(\varrho)^{\mu}, \qquad \mu\doteq-\frac{mp-3p+2}{p-1},$$

which is shown to satisfy [2, (2.45)] for large R, namely,

$$\frac{\log I(R)}{R^2}\ge\frac{\log I(R_0)}{R^2}+S\frac{(1+2\delta)}{c_1} \tag{0.3}$$

for positive constants R_0,S,δ,c_1. The gap in the proof lies in the subsequent argument, where we show that $R^{-2}\log I(R)\to 0$ as $R\to\infty$ and thus contradict (0.3). The issue is that ϱ is not a priori controlled by the distance to a fixed point, so comparison theorems are not applicable. Here is the modified argument. First, observe that if we can prove that

$$\liminf_{R\to\infty}\frac{\log\|h(\varrho)^{\mu}\|_{L^\infty(\Omega\cap B_R)}}{R^2}=0, \tag{0.4}$$

then we reach a contradiction as follows: by (0.1) (i) and Bishop-Gromov comparison theorem, recalling that $0\le\lambda\le 1$ we get

$$\log I(R)\le\log|B_R|+\log\|h(\varrho)^{\mu}\|_{L^\infty(\Omega\cap B_R)}\le C_1+C_2R+\log\|h(\varrho)^{\mu}\|_{L^\infty(\Omega\cap B_R)}$$

for suitable constants C_j, so we can let $R\to\infty$ in (0.3) along a sequence realizing the liminf and conclude $0\ge S(1+2\delta)/c_1$, which is absurd.

Case (a). Assumption $m=2$, $p\ge 2$ is equivalent to $\mu\ge 0$. If $\mu=0$ then (0.4) trivially holds. If $\mu>0$, $h(\varrho)^{\mu}$ is unbounded when $\varrho(x)\to\infty$, that is, when $u(x)\to 0$. The conclusion will follow from (0.4) via an approximation argument.

We let $c>0$ and consider the function ϱ_c defined by the identity

$$u(x)+c=\int_{\varrho_c(x)}^{\infty} v_h(s)^{-\frac{1}{p-1}}\mathrm{d}s \quad \text{on } \Omega_c \doteq \left\{x\in\Omega : u(x)+c<\mathscr{G}^h(0)\right\}.$$

Notice that $\Omega_c\uparrow\Omega$ and $\varrho_c\uparrow\varrho$ pointwise in Ω as $c\downarrow 0$, whence

$$\mathrm{Ric}\geq -(m-1)H(\varrho)\geq -(m-1)H(\varrho_c) \quad \text{on } \Omega_c,$$

and moreover

$$|\nabla\varrho_c|=|\nabla u|v_h(\varrho_c)^{\frac{1}{p-1}}\leq |\nabla u|v_h(\varrho)^{\frac{1}{p-1}}=|\nabla\varrho|. \tag{0.5}$$

Also, $\varrho_c\to\varrho$ locally in $C^1(\Omega)$, as can be argued by differentiating the very definitions of ϱ_c and ϱ. By construction, ϱ_c is bounded, whence (0.4) holds for $\varrho=\varrho_c$ and we can therefore deduce from the integral estimates leading to (0.3) with ϱ_c replacing ϱ the inequality

$$\sup_{\Omega_c}|\nabla\varrho_c|\leq \max\left\{1, \limsup_{\Omega_c\ni y\to\partial\Omega_c}|\nabla\varrho_c(y)|\right\}. \tag{0.6}$$

However, if $y\in\partial\Omega_c\cap\Omega$ then $\varrho_c(y)=0$, and therefore $\nabla\varrho_c(y)=0$ by the first equality in (0.5). Again by (0.5) we conclude

$$\limsup_{\Omega_c\ni y\to\partial\Omega_c}|\nabla\varrho_c(y)|=\begin{cases}\limsup\limits_{\Omega_c\ni y\to\partial\Omega}|\nabla\varrho_c(y)|\leq\limsup\limits_{y\to\partial\Omega}|\nabla\varrho(y)| & \text{if } \partial\Omega\neq\emptyset\\ 0 & \text{if } \Omega=M.\end{cases}$$

Inserting the latter into (0.6), observing that $\varrho_c\to\varrho$ in $C^1_{\mathrm{loc}}(\Omega)$ guarantees that $|\nabla\varrho_c(x)|\to|\nabla\varrho(x)|$ for each $x\in\Omega$, and letting $c\to 0$ we get the desired (0.2) (with $|\nabla\varrho|\leq 1$ if $\Omega=M$).

Case (b). Since (a) fails, in this case $\mu<0$. Hence, to estimate $h(\varrho)^\mu$ we need to consider its behavior as $\varrho(x)\to 0$, that is, as $u(x)\to+\infty$. If $p>m$, our L^∞ condition on u implies that ϱ is bounded below by a positive constant, whence $h(\varrho)^\mu\in L^\infty(\Omega)$ and (0.4) holds. If $p\leq m$, we know that $h(t)\sim t$ and $\mathscr{G}^h(t)\sim\mu(t)$ as $t\to 0$, where $\mu(t)$ is as in [2, (2.5)]. Hence, there exist constants $C_j=C_j(m,p,H)$ such that

$$\|h(\varrho)^\mu\|_{L^\infty(\Omega\cap B_R)}\leq\begin{cases}C_1+C_2\|u\|_{L^\infty(\Omega\cap B_R)}^{\frac{|\mu|(p-1)}{m-p}} & \text{if } p<m\\ C_1+\exp\left\{C_2\|u\|_{L^\infty(\Omega\cap B_R)}\right\} & \text{if } p=m,\end{cases}$$

and (0.4) follows by the growth conditions in (b).

The final part of the proof regards the case $H_*=0$, and no modification is needed, up to choosing the approximation parameter c small enough.

The change in Lemma 2.17 only affects the statement of [2, Thm. 2.22]. Indeed, in Theorem 2.24 and Remark 4.9 one deals with $u \in L^\infty(\Omega)$, so the amended version of the lemma suffices. In Theorem 2.19, the lemma shall be applied on $\Omega \doteq M \backslash \overline{B_\varepsilon}$ rather than on $M \backslash \{o\}$ (B_ε a small ball centered at o), and then one lets $\varepsilon \to 0$. Theorem 2.22 in [2] is to be modified as follows:

THEOREM 0.3. *Let M^m be complete, let $\Omega \subset M$ be an open set (possibly the entire M) and suppose that*

$$\inf_M \mathrm{Ric} > -\infty, \quad \mathrm{Ric} \geq -(m-1)\kappa^2 \quad \text{on } \Omega, \tag{0.7}$$

for some constant $\kappa \in \mathbb{R}^+_0$. Let $u > 0$ solve $\Delta_p u = 0$ on Ω for some $p \in (1,\infty)$, and assume that either $m = 2 \leq p$ or that one of the following conditions holds for some sequence $R_j \to \infty$:

$$\begin{aligned} \log \|u\|_{L^\infty(\Omega \cap B_{R_j})} &= o(R_j^2) \quad \text{if } p < m, \\ \|u\|_{L^\infty(\Omega \cap B_{R_j})} &= o(R_j^2) \quad \text{if } p = m, \\ u &\in L^\infty(\Omega) \quad \text{if } p > m. \end{aligned}$$

Then,

$$|\nabla \log u| \leq \max\left\{ \frac{m-1}{p-1}\kappa, \ \limsup_{x \to \partial\Omega} |\nabla \log u| \right\}.$$

The proof in [2] rests on applying Lemma 2.17 to the function $cu(x)$ and taking limits as $c \to 0$. Notice that, when $p > m$, (b) is satisfied for c small enough.

Remark 0.4. Theorem 0.3 for $\Omega = M$ does not allow to recover the main Theorem in [3] in its full strength any more. Still, we feel it of some interest in view of the possibly non-empty boundary, a case for which the proof in [3] seems, to us, not easy to adapt.

Further changes.

LEMMA 3.3. *For $q \in (0,p)$, the half-Harnack inequality (3.12) in* [2] *should be restricted to solutions $u > 0$ to $\Delta_p u = 0$, so as to guarantee the applicability of* [2, Lemma 3.2] *to each exponent $\bar{q} = q_i$ in Moser's iteration. On the other hand, if $q \geq p$ the problem does not occur, and the half-Harnack inequality holds for solutions to $\Delta_p u \geq 0$ as stated. Applications are not affected: indeed, (3.12) is applied to p-harmonic functions in Theorem 3.4, while in Theorem 3.6 it is used with the choice $q = \nu p/(\nu - p) \geq p$.*

PROPOSITION 4.3. *In the nondegeneracy inequality* [2, (4.8)]*, we should assume that $B_R \Subset \Omega$, so that* $\min_{\partial B_R} \varrho_1$ *replaces the liminf. The change serves to guarantee that "By local uniform convergence, for $p \in \{p_j\}$ close enough to* 1*, $\varrho_p > \tau'$ on $\Omega \cap \partial B_R$"* [2, two lines before (4.9)]. *While the sentence might be true* *in the original assumptions, because of a possible convergence issue for $\varrho_p \to \varrho_1$ near $\partial\Omega \cap \partial B_R$ we settle for a slightly weaker version of the Lemma, which however suffices for applications to Theorems 4.4 and 4.6.*

LEMMA 5.3. *The words "on an open dense subset of $\mathbb{R}^+$" should be replaced by "almost everywhere on $\mathbb{R}^+$": indeed, (i), (ii) hold for a.e. Lebesgue point of both $\mathscr{V}_u$ and $\mathscr{A}_u$. A related requirement on t_i (last line of p. 845) is not needed and should be removed.*

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